e

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What is this sequence approaching?

1^1

1.1^{10}

1.01^{100}

1.001^{1000}

1.0001^{10000}

$\dots$

It is approaching $e = 2.718\ldots$

$$1^1 = 1$$

$$1.1^{10} = 2.593\ldots$$

$$1.01^{100} = 2.704\ldots$$

$$1.001^{1000} = 2.716\ldots$$

$$1.0001^{10000} = 2.718\ldots$$

$\ldots$

A more general formula

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In general,

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

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- Compounded continuously:

$$1000e^{.05} = \$1419.07$$

Infinite series

From Taylor series or binomial theorem on $(1+\frac{1}{n})^n$:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

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In general, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Six lines of Python to compute 1000 digits of e

```
from fractions import Fraction
from math import factorial
from decimal import Decimal, getcontext

getcontext().prec = 1000
x = sum(Fraction(1,factorial(i)) for i in range(450))
e = Decimal(x.numerator) / Decimal(x.denominator)
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By Taylor's theorem, error is less than $\frac{e}{451!} = 3.5 \times 10^{-1003}$.

2.718281828459045235360287471352662497757247093699959574966967627
7240766303535475945713821785251664274274663919320030599218174135
9662904357290033429526059563073813232862794349076323382988075319
5251019011573834187930702154089149934884167509244761460668082264
8001684774118537423454424371075390777449920695517027618386062613
3138458300075204493382656029760673711320070932870912744374704723
0696977209310141692836819025515108657463772111252389784425056953
6967707854499699679468644549059879316368892300987931277361782154
2499922957635148220826989519366803318252886939849646510582093923
9829488793320362509443117301238197068416140397019837679320683282
3764648042953118023287825098194558153017567173613320698112509961
8188159304169035159888851934580727386673858942287922849989208680
5825749279610484198444363463244968487560233624827041978623209002
1609902353043699418491463140934317381436405462531520961836908887
0701676839642437814059271456354906130310720851038375051011574770
41718986106873969655212671546889570350354...

Computed digits of e (from Wikipedia)

Number of known decimal digits of *e*

Date	Decimal digits	Computation performed by
1748	23	Leonhard Euler[19]
1853	137	William Shanks
1871	205	William Shanks
1884	346	J. Marcus Boorman
1949	2,010	John von Neumann (on the ENIAC)
1961	100,265	Daniel Shanks and John Wrench[20]
1978	116,000	Stephen Gary Wozniak (on the Apple II[21])
1994 April 1	1,000,000	Robert Nemiroff & Jerry Bonnell [22]
1999 November 21	1,250,000,000	Xavier Gourdon [23]
2000 July 16	3,221,225,472	Colin Martin & Xavier Gourdon [24]
2003 September 18	50,100,000,000	Shigeru Kondo & Xavier Gourdon [25]
2007 April 27	100,000,000,000	Shigeru Kondo & Steve Pagliarulo [26]
2009 May 6	200,000,000,000	Rajesh Bohara & Steve Pagliarulo [26]
2010 July 5	1,000,000,000,000	Shigeru Kondo & Alexander J. Yee [27]

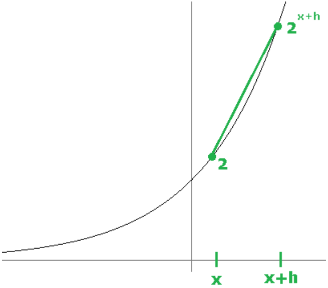
Derivatives

- e^x is best known for being its own derivative.
- It is essentially the only function with that property.
- Why?

Derivative of $y = 2^x$

Derivative is the slope of the tangent line.

Approximate it by secant lines.



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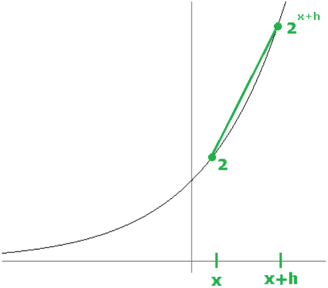

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

As $h \to 0$, we approach the slope of the tangent line.

Derivatives

$$\text{slope} = \lim_{h \to 0} \frac{2^h - 1}{h} 2^x$$

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- $\displaystyle \lim_{h \to 0} \frac{2^h - 1}{h} \approx .693$
- Try 3^x: $\displaystyle \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.099$
- Try 4^x: $\displaystyle \lim_{h \to 0} \frac{4^h - 1}{h} \approx 1.386$

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- Getting worse...we want a limit of 1. Need a power between 2 and 3...
- The power that works is e^x. Namely, $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

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- $e \approx (1 + h)^{1/h}$
- So $e^h \approx 1 + h$.
- And $\frac{e^h - 1}{h} \approx 1$.

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Logarithms

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x	$y = \log_{10} x$
1	0
10	1
100	2
1000	3
10000	4

A multiplicative change in x corresponds to an additive change in y.

Formally,

$$\log(ab) = \log(a) + \log(b)$$

$\int x^n \, dx = \frac{x^{n+1}}{n+1}$. But what if $n = -1$?

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Logarithms and Integrals

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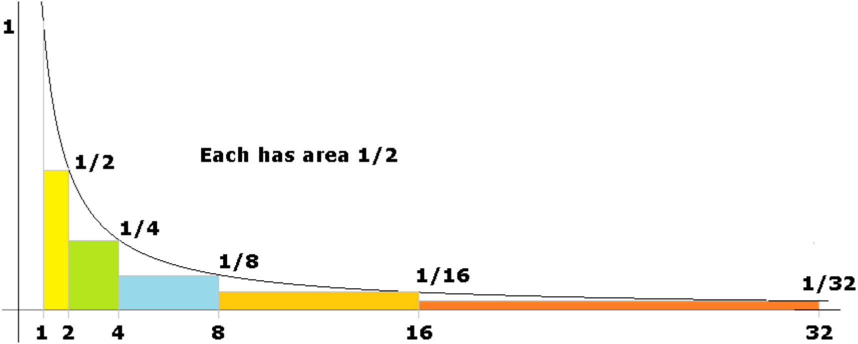

A multiplicative change in x corresponds to an additive change in the total area.

What base?

This leads to

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But why not something else, like base 7 or base 443.18?

Why base e

Say we want $\displaystyle\int_1^{32} \frac{1}{x}\,dx$.

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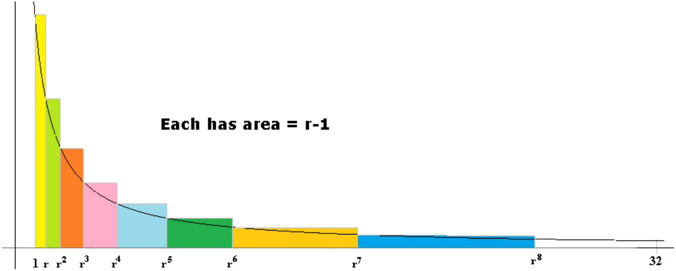
Suppose instead of powers of 2, we use something smaller, like powers of $r = 1.5$.

The smaller rectangles will fit the area more closely.



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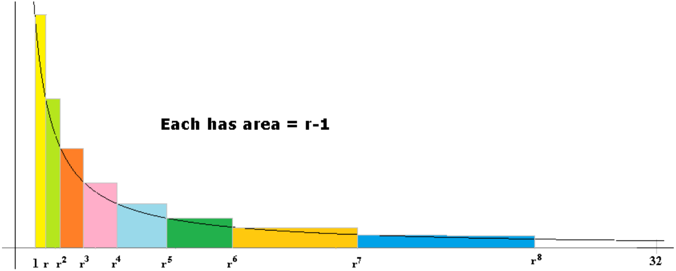

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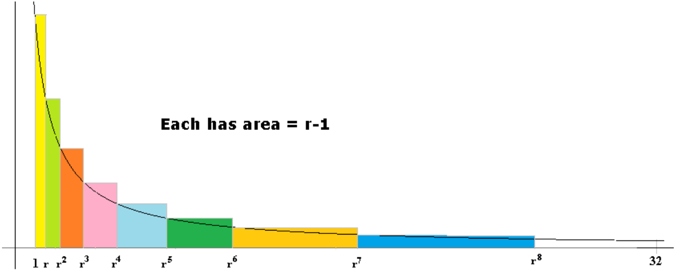

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Answer: Find the largest power of r less than 32.

In other words, solve $r^x = 32$. We get $x = \dfrac{\log(32)}{\log r}$.

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The area is then $\frac{\log(32)}{\log r}(r-1)$.

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The area is then

$$\begin{aligned} &\frac{\log(32)}{\log(1+\frac{1}{n})}(1+\frac{1}{n}-1) \\ =&\frac{\log(32)}{n\log(1+\frac{1}{n})} \\ =&\frac{\log(32)}{\log(1+\frac{1}{n})^n} \\ =&\log_{(1+\frac{1}{n})^n}(32) \end{aligned}$$

As $n \to \infty$, this becomes $\log_e(32)$.

e in calculus

In summary, e is so important in calculus because:

- $f(x) = e^x$ is (more or less) the only function whose derivative is itself
- $\log_e(x)$ is the antiderivative of $\frac{1}{x}$.

Euler's formula

e^x is directly related to $\sin x$ and $\cos x$:

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If you plug in $\theta = \pi$, you get...

"The most remarkable formula in mathematics"

$$e^{i\pi} + 1 = 0$$

e is irrational

Proof: Suppose $e = \frac{p}{q}$. Using the power series for e, we have

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!} + \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \ldots$$

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Multiply both sides by $q!$ to get

$$p(q-1)! = q! + q! + q(q-1)\ldots 3 \cdot 2 + \cdots + 1 + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots$$

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The left side is an integer. The right side is not because $\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots < 1$.

Contradiction!

If you want the details...

Even if $q = 2$, we have $\frac{1}{3} + \frac{1}{12} + \frac{1}{60} + \ldots$, which is small.

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Formally,

$$\begin{aligned}
&\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \dots \\
\leq &\frac{1}{2+1} + \frac{1}{(2+1)(2+2)} + \dots \\
< &\frac{1}{3} + \frac{1}{3^2} + \dots \\
= &\frac{1}{1-\frac{1}{3}} - 1 \\
= &\frac{1}{2}
\end{aligned}$$

What is the shape of a wire hanging between two points?

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It's called a *catenary* and its equation is $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$.

A famous Catenary

(Gateway Arch in St. Louis)

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How many ways are there to rearrange so that no number stays fixed?

This is called a *derangement*.

A curious number

n	Derangements (d_n)	Rearrangements (r_n)	d_n/r_n
1	0	1	0.000000
2	1	2	0.500000
3	2	6	0.333333
4	9	24	0.375000
5	44	120	0.366667
6	265	720	0.368155
7	1854	5040	0.367857
8	14833	40320	0.367881
9	133496	362880	0.367879
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$$\frac{1}{e}$$

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- How many derangements of 1, 2, 3, 4, 5 are there?

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- There are 5! rearrangements. From these subtract off the number that contain fixed elements.
- Number of rearrangements that fix one element: 5 choices for the element, 4! ways to arrange others, so $5 \cdot 4!$ in total.

Derangements, continued

- How many derangements of 1, 2, 3, 4, 5 are there?
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- Etc. Approach is called inclusion-exclusion.

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- In general: $n!(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!})$

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- We see the power series for $1/e$ here

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- Euler did a lot with e.

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- Google's 2004 IPO announced they were trying to raise \$2,718,281,828.

Thanks!

Thank you for your attention.