Beautiful Images from Some Simple Formulas

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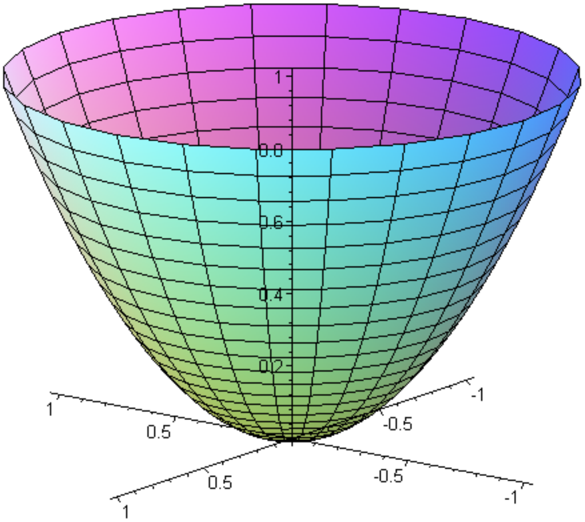
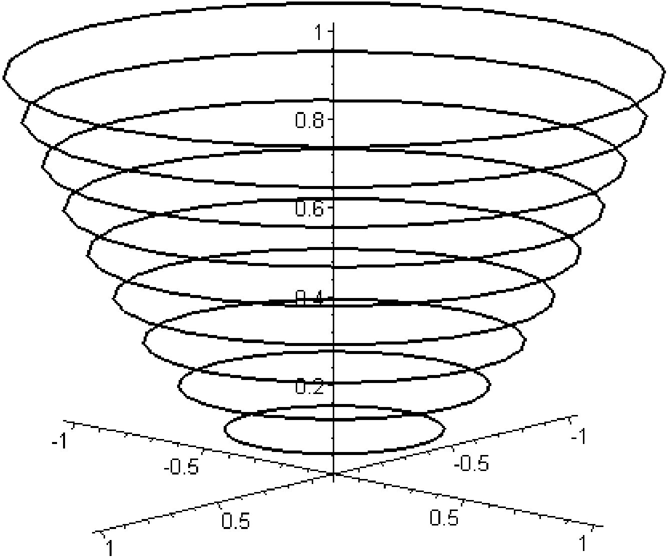
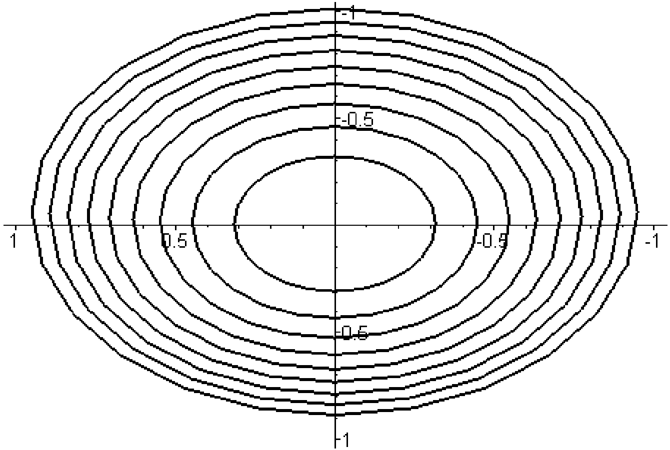
Ordinary plot of $f(x, y) = x^2 + y^2$



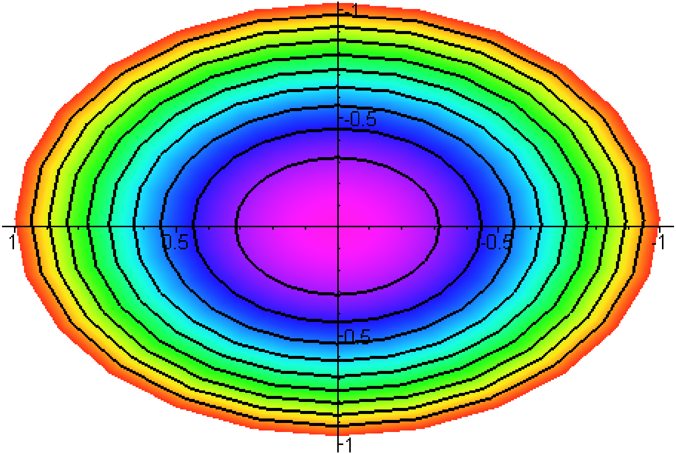
Contour plot of $f(x, y) = x^2 + y^2$



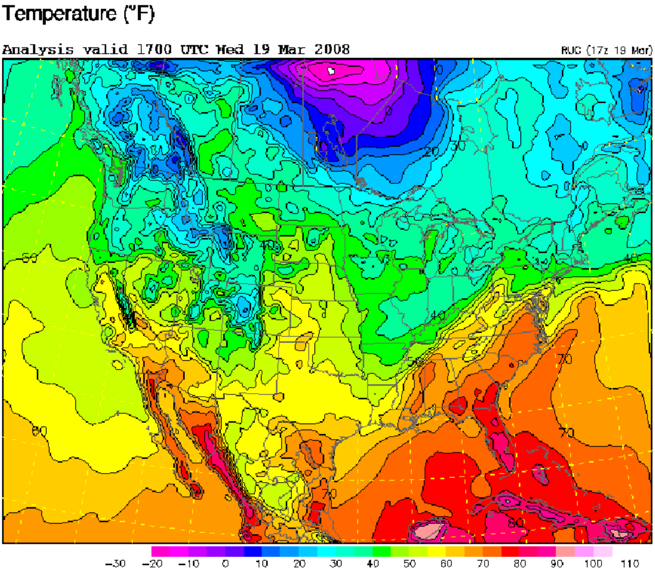
Looking down from above



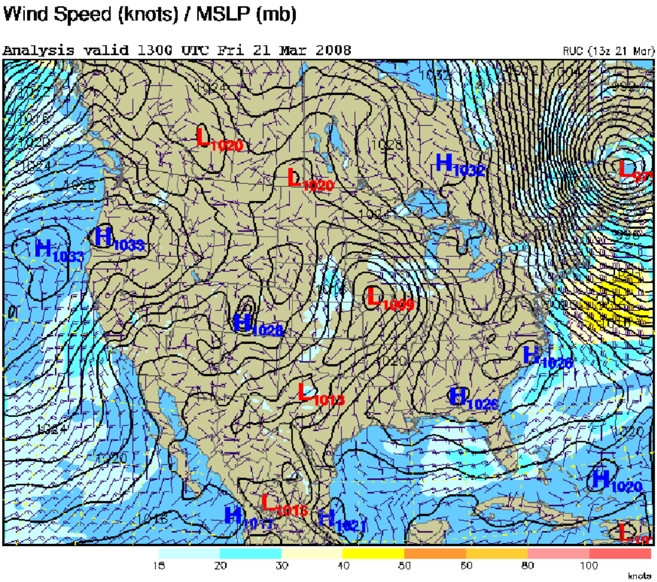
Colored contour map



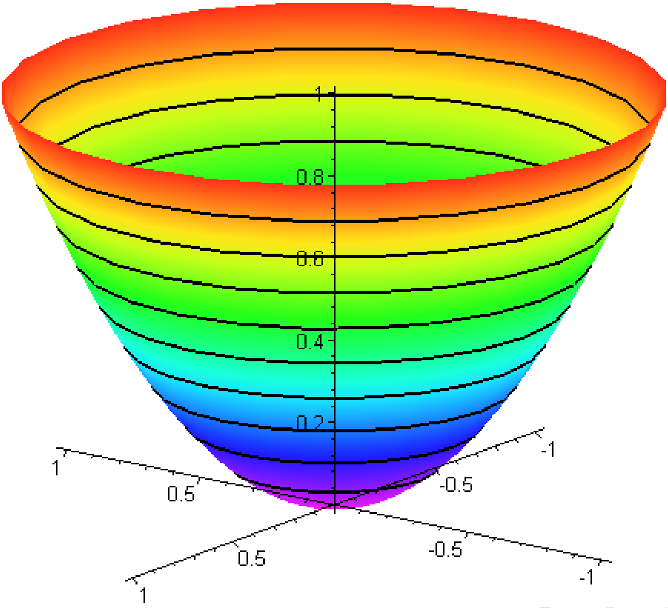
Temperatures maps are contour maps.



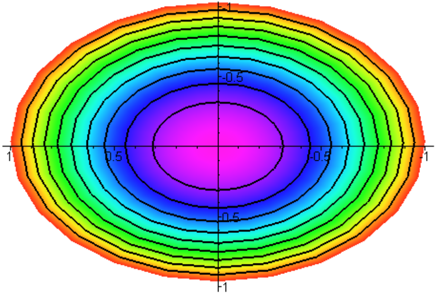
Pressure maps are, too.

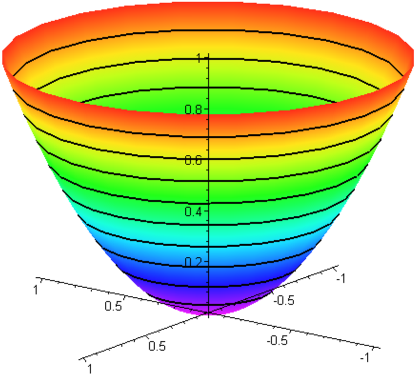


Back to $f(x, y) = x^2 + y^2$

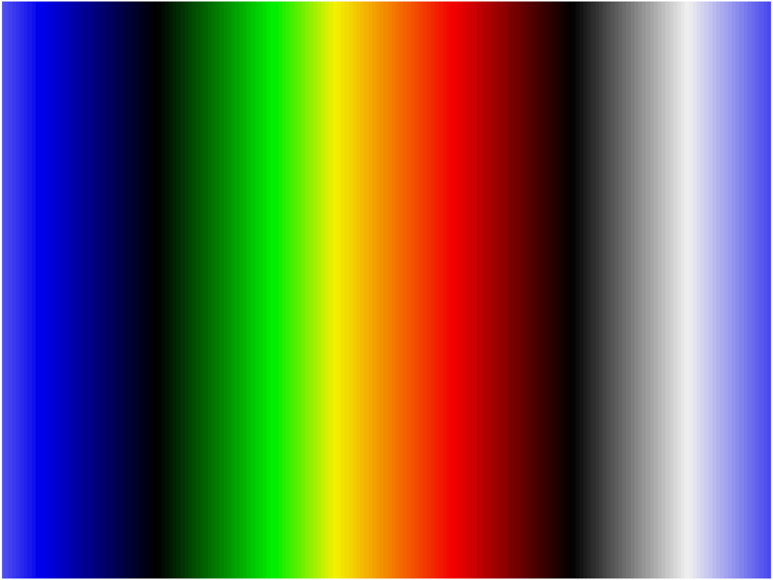


Graph vs. contour plot

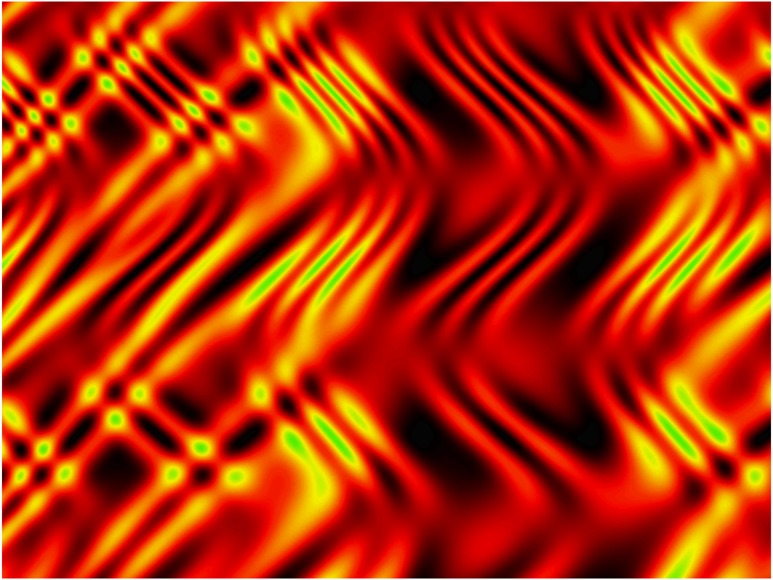




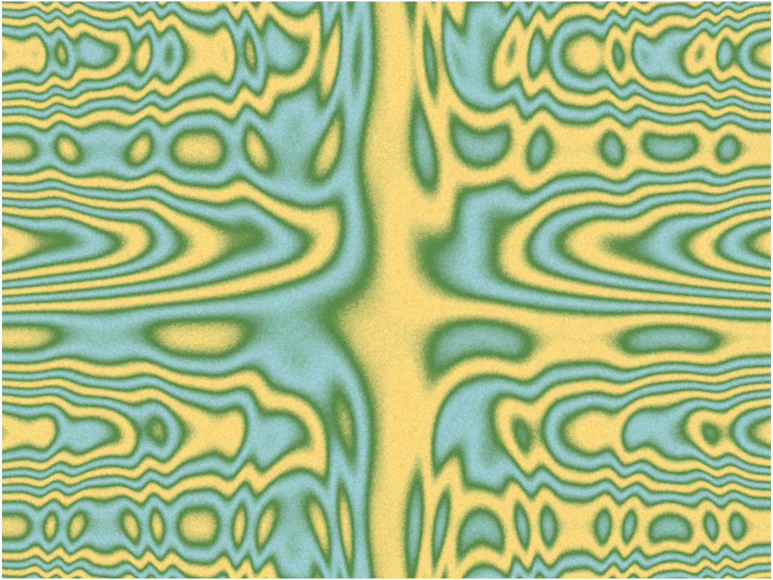
Color scheme



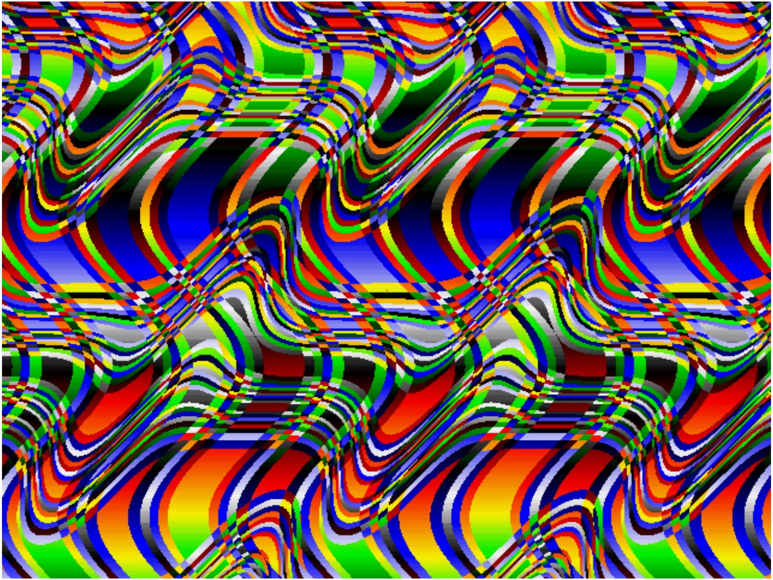
$\cos(x \sin(y - x) + \cos(y)) + \sin(y) \cos(x - y \sin(x - \cos(y)))$



$\sin(x + \cos(y - y\cos(x)) + x\sin(y)) + \mathrm{rand}(100)/400$



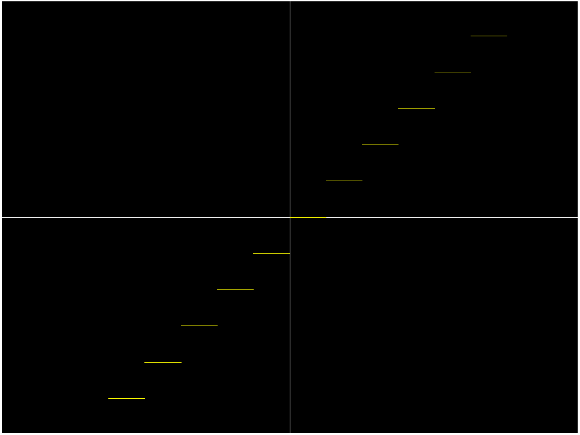
$$\mathrm{fact}(\mathrm{abs}(\mathrm{floor}(12\cos(\sin(x+\sin(y))))))$$
$$+\,\mathrm{fact}(\mathrm{abs}(\mathrm{floor}(11\sin(y+\cos(x+\sin(y-x)))))) + y$$



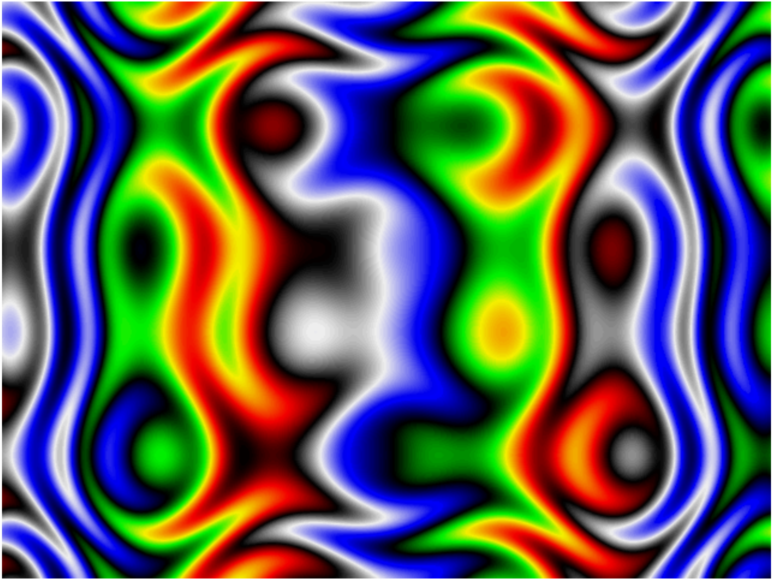
Floor function

The floor function returns the greatest integer less than or equal to the given number.

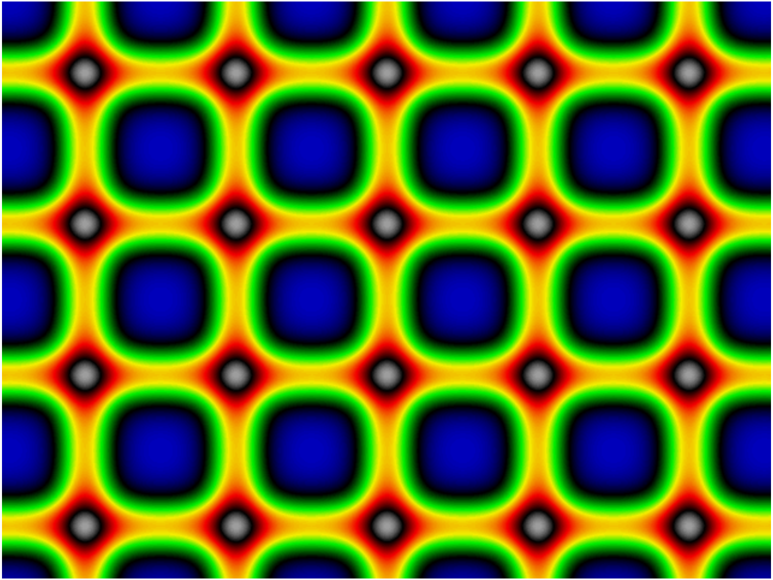
$\text{floor}(2.56) = 2$, $\text{floor}(3.98) = 3$, and $\text{floor}(-3.98) = -4$



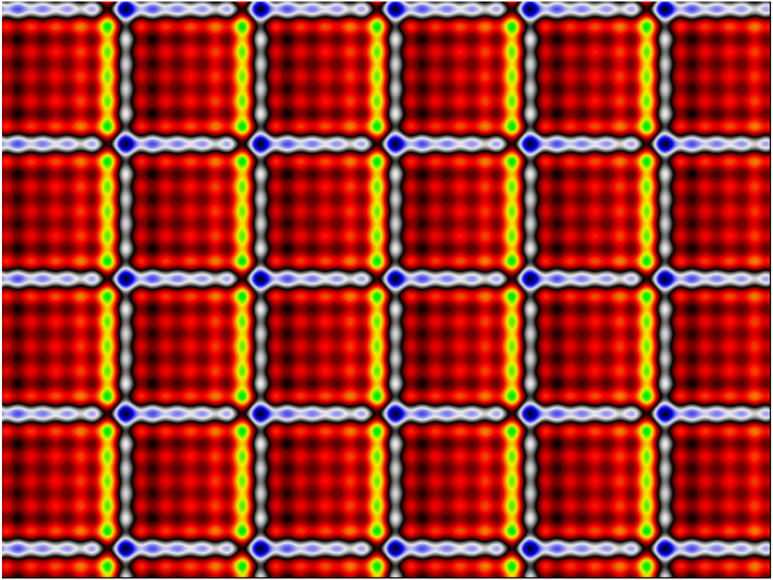
Another complicated sines and cosines formula



$2^{2\cos(x)} + 2^{2\sin(y)}$

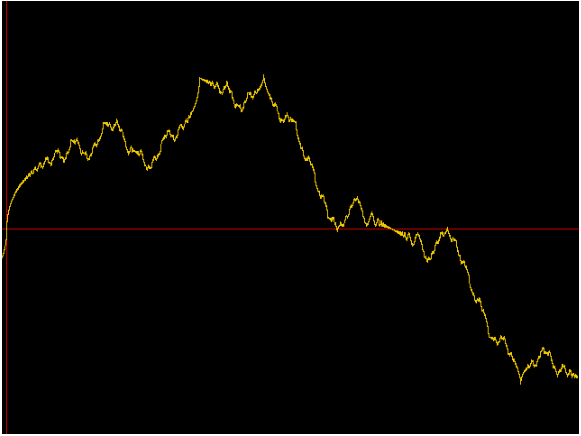


$\sin(x) + \cos(y) + \sin(2x) + \cos(2y) + \sin(3x) + \cos(3y)$
$+ \sin(4x) + \cos(4y) + \sin(5x) + \cos(5y)$



Riemann function

$$\text{riem}(x) = \sum_{n=1}^{\infty} \frac{\sin\left(n^2 x\right)}{n^2}$$



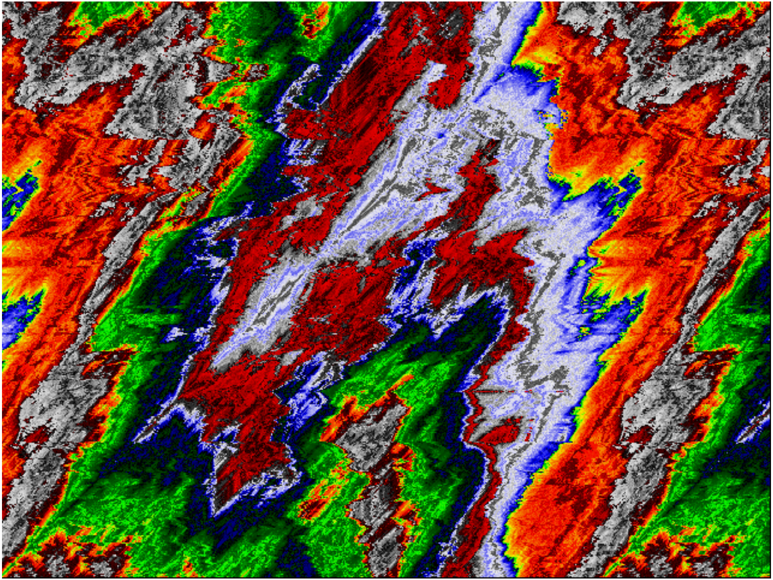
It is continuous everywhere, but differentiable almost nowhere.

The best we can do is approximate it:

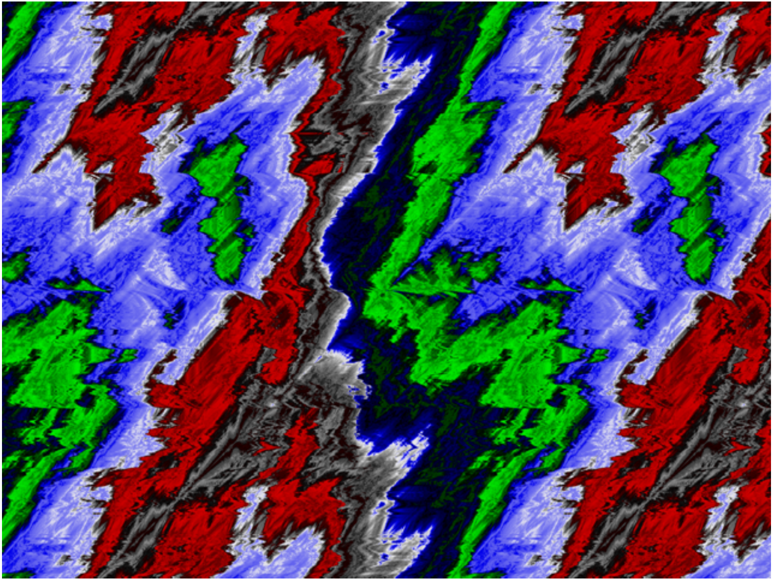
$$\operatorname{riem}(x,k) = \sum_{n=1}^{k} \frac{\sin\left(n^2 x\right)}{n^2}$$

$$\operatorname{riemc}(x,k) = \sum_{n=1}^{k} \frac{\cos\left(n^2 x\right)}{n^2}$$

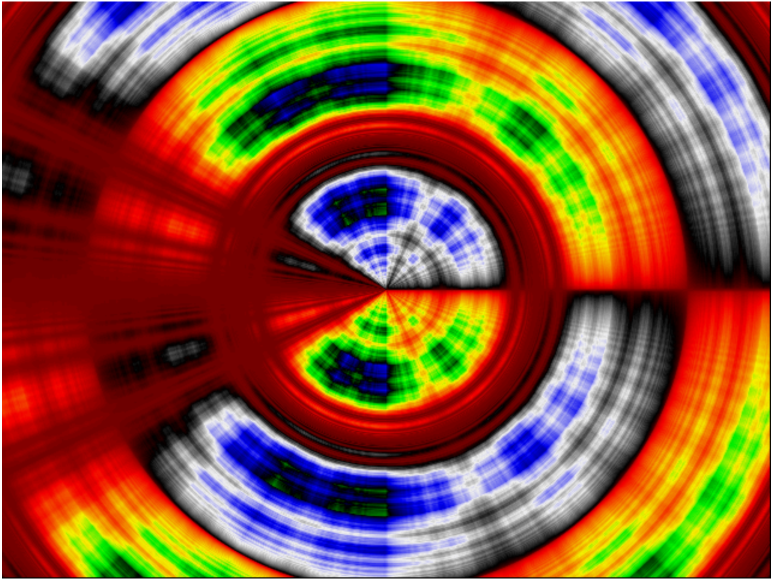
$$\text{riem}(x + \text{riemc}(y - x) + \text{riem}(\text{riemc}(x + \text{riem}(y)) + x))$$



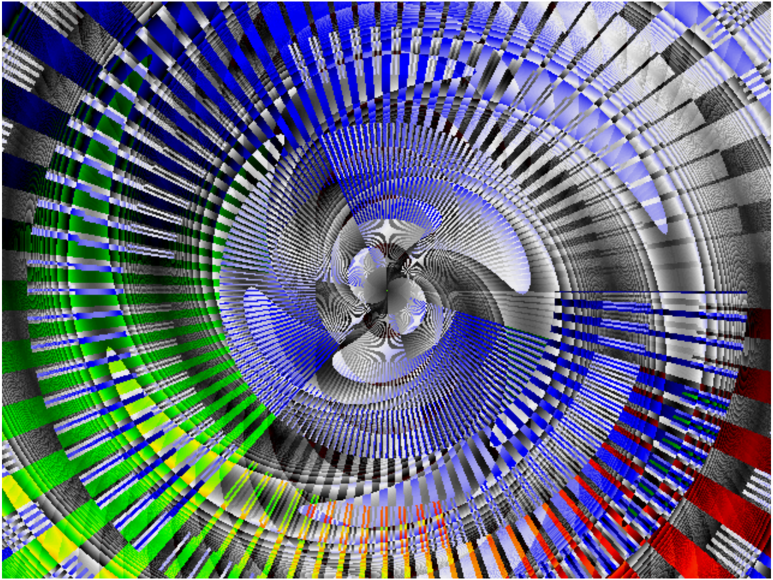
$\mathrm{riem}(x + \mathrm{riemc}(y - x) + \mathrm{riem}(\mathrm{riemc}(x + \mathrm{riem}(y)) + x))$



$\mathrm{riem}(t, 20)\,\mathrm{riem}(r, 20)$



More polar coordinates



Mod function

It returns the remainder when a number is divided by another.

$20 \bmod 7 = 6$ because the remainder when 20 is divided by 7 is 6.

It is represented by % in the formulas.

Bitwise AND function

We represent it by the symbol &

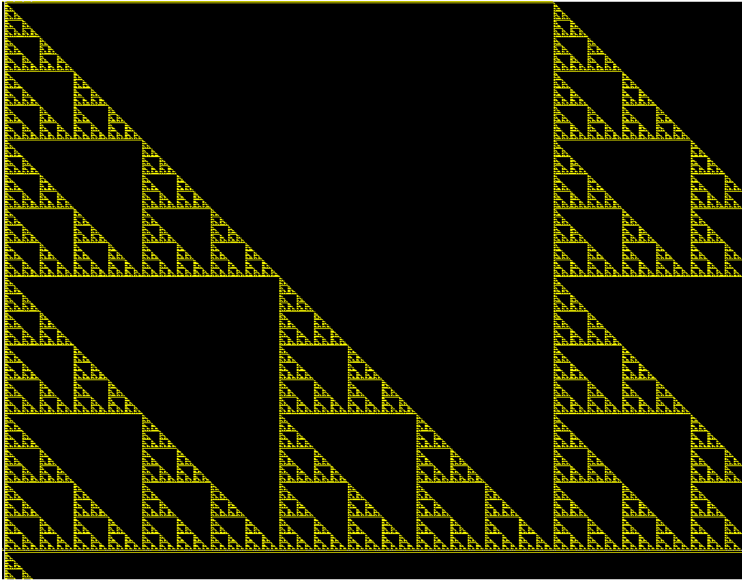
1=True, 0 = False

$$1 \,\&\, 1 = 1, \qquad 1 \,\&\, 0 = 0, \qquad 0 \,\&\, 1 = 0, \qquad 0 \,\&\, 0 = 0$$

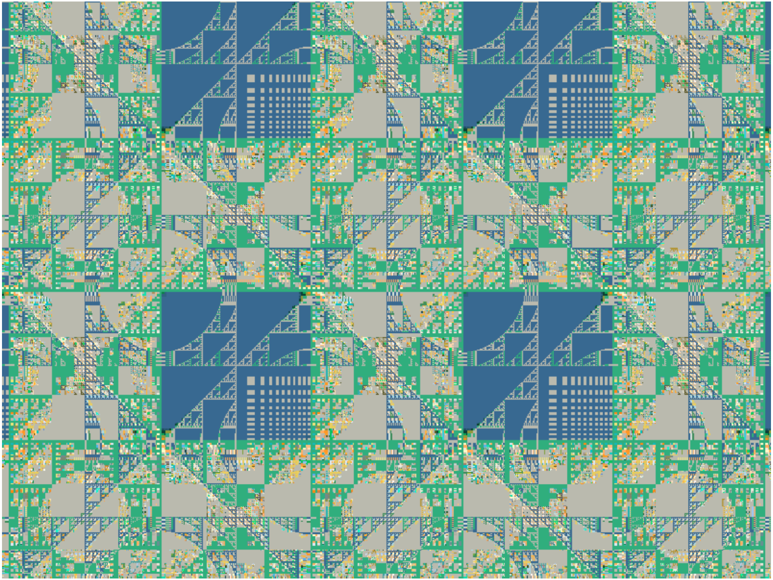
To compute 11&14:

1. Convert each to binary $\rightarrow 1011 \,\&\, 1110$
2. AND the corresponding digits $\rightarrow 1010$
3. Convert back to decimal $\rightarrow 10$

Plot of $x \,\&\, y = 0$



$\sqrt{\tan(100\sin(x)\&100\cos(y))^{20\cos(x)\&20\sin(y)}}$



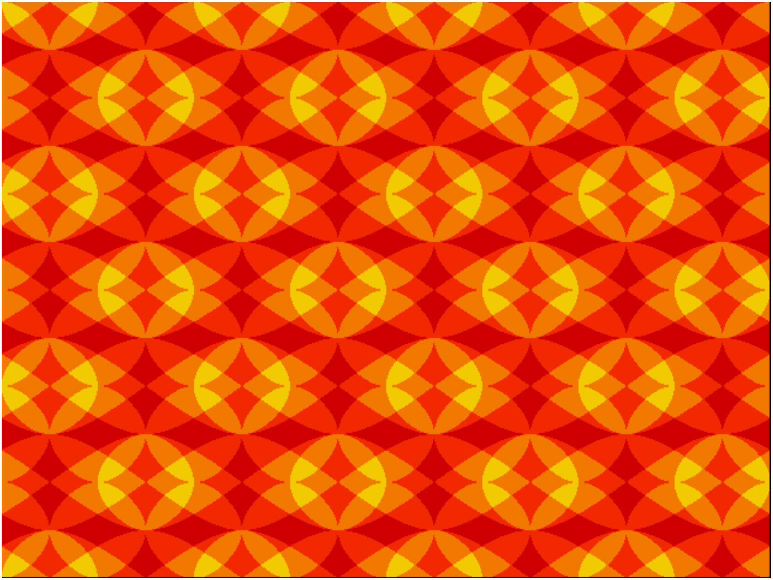
Bitwise NOT function

It is the logical not function, represented by !.

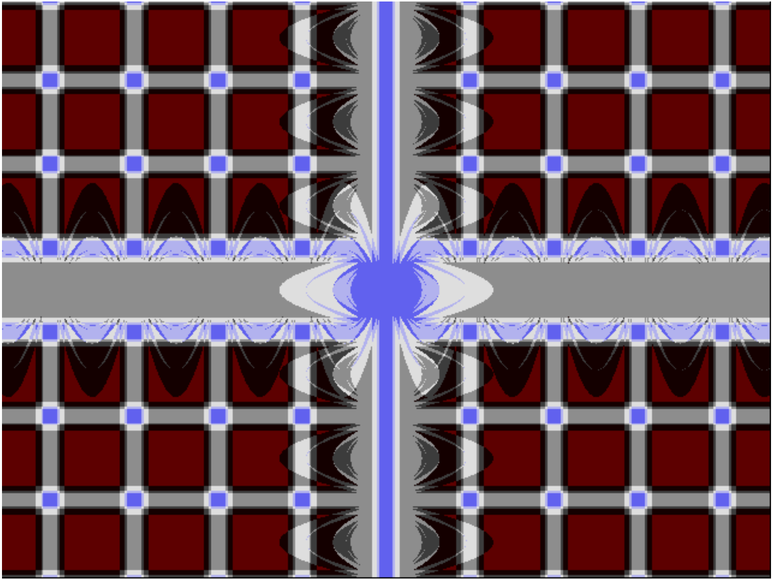
$!1 = 0$ and $!0 = 1$

Extend this to $\mathbb{R}$ by defining $!x$ to equal 1 if $-1 < x < 1$ and 0 otherwise.

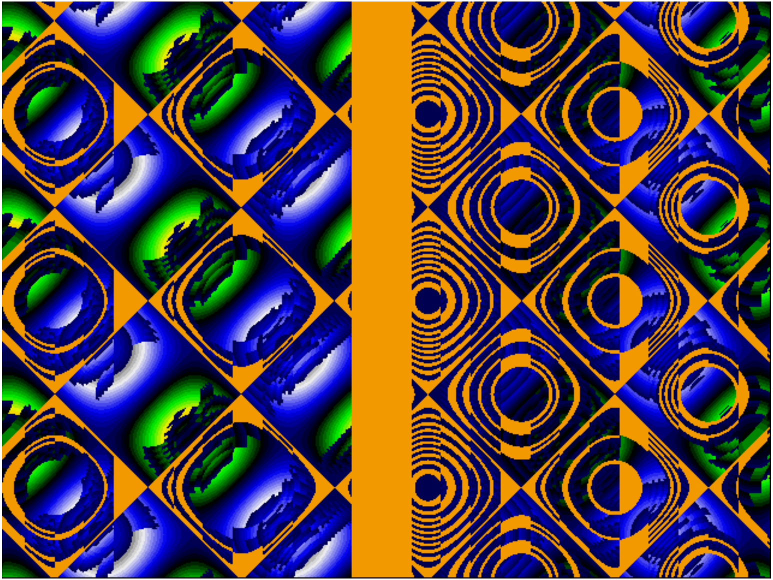
$$!(\cos(x)+\sin(y)) + !(\cos(x)-\sin(y)) + !(\sin(x)-\cos(y)) + !(\sin(x)-2\cos(y)) + !(\sin(x)+2\cos(y))$$



$!(2\cos(y)\%x)+!(2\sin(x)\%y)+!(3\cos(y)\%x)+$ $!(3\sin(x)\%y)+!(4\cos(y)\%x)+!(4\sin(x)\%y)$



$\text{floor}(10(\cos(x) + \sin(y)))\%(x\&\,\text{floor}(10(\sin(x) + \cos(y))))$



Complex numbers

$i = \sqrt{-1}$ (solution to $x^2 + 1 = 0$)

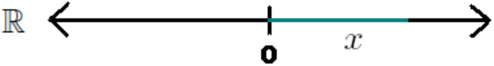
Examples: $2i, \quad 3 + 4i, \quad -.2 + .76i$

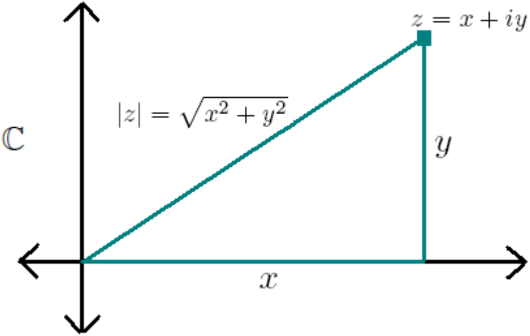
Addition: $(2 + 3i) + (5 + 8i) = 7 + 11i$

Multiplication: $(2 + 3i)(5 + 8i) = 10 + 31i + 24i^2 = -14 + 31i$

Division: $\frac{2+3i}{5+8i} = \frac{2+3i}{5+8i} \cdot \frac{5-8i}{5-8i} = \frac{34-8i}{89} = \frac{34}{89} + \frac{8}{89}i$

Picturing them





Example: Let $f(x) = x^2$ and start with $x = 2$.

$f(2) = 4$
$f(4) = 16$
$f(16) = 256$
$f(256) = 65536$
$\dots$

Iterates are approaching ∞.

A different starting point

Let $f(x) = x^2$ and start with $x = \frac{1}{2}$.

$f(\frac{1}{2}) = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{16}$

$f(\frac{1}{16}) = \frac{1}{256}$

$f(\frac{1}{256}) = \frac{1}{65536}$

$\dots$

Iterates are approaching 0.

Another example

Let $f(x) = -x$ and start with $x = 1$.

$f(1) = -1$
$f(-1) = 1$
$f(1) = -1$
$f(-1) = 1$
$\dots$

Iterates are not settling down on a value.

Color each point according to how fast it converges.

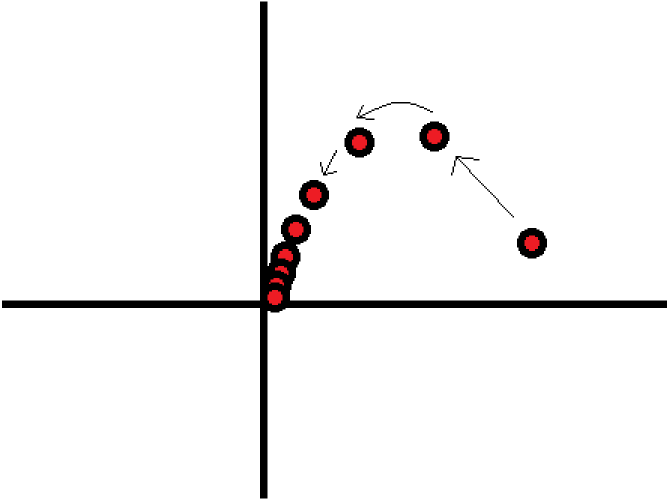
Count how many iterations until two successive values are within .00001 of each other.

Assign each count a color.

Convergence to infinity is still convergence (color by # of steps to exceed $\pm 10^5$).

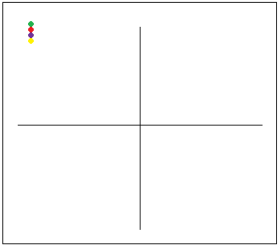
Iteration with complex numbers

Plug $z = x + iy$ into $f(z)$. Get a value, and plug that value into the function. Then plug the result of that into the function, etc.



The process

Look at all the possible starting values in a region.

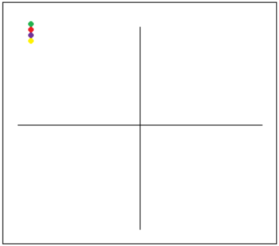


For each starting point, iterate the function.

If two successive values are within .00001 of each other, there's a very good chance that the iterates will converge.

The process, continued

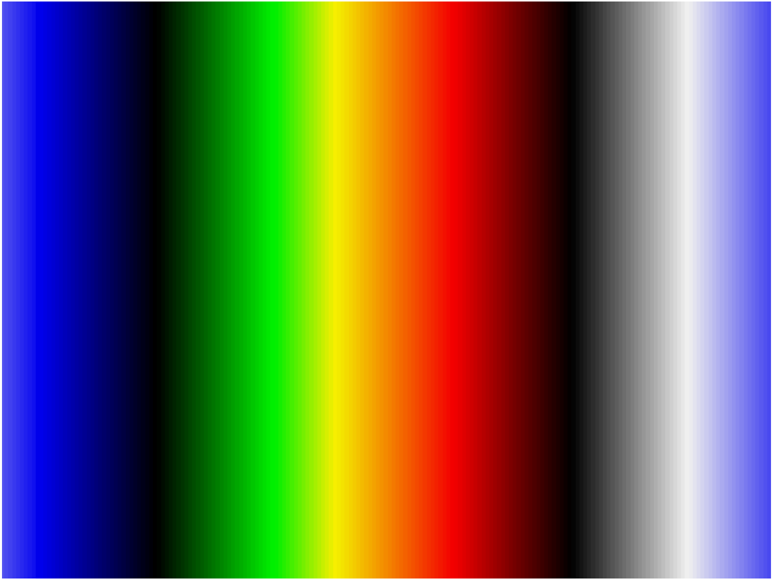
In this case, color the point with a color representing how long it took for this to happen.



It is possible that the iteration may never stop. Give up after a few hundred iterations and color the point yellow.

Note: convergence to infinity is still convergence (color by how many steps for iteration to exceed $\pm 10^5$).

Color scheme



$f(z) = c \cdot \sin(\ln z)$

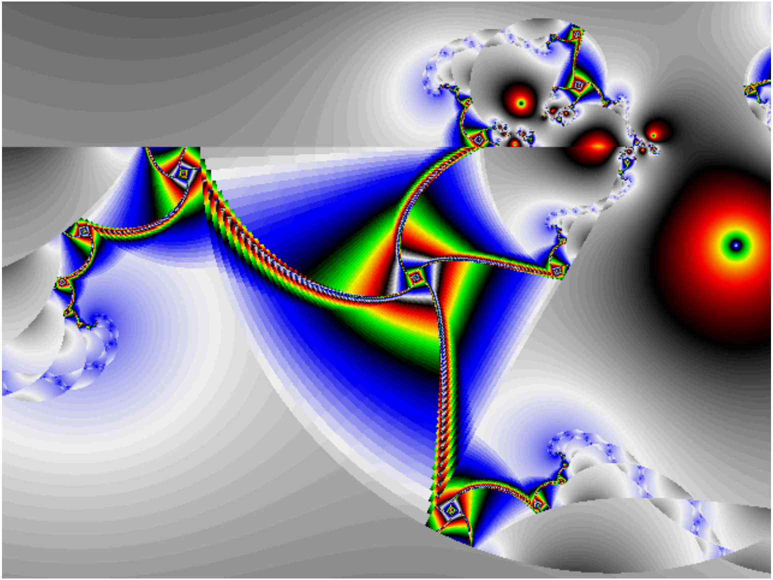
$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

$$\ln z = \ln|z| + i \arg z$$

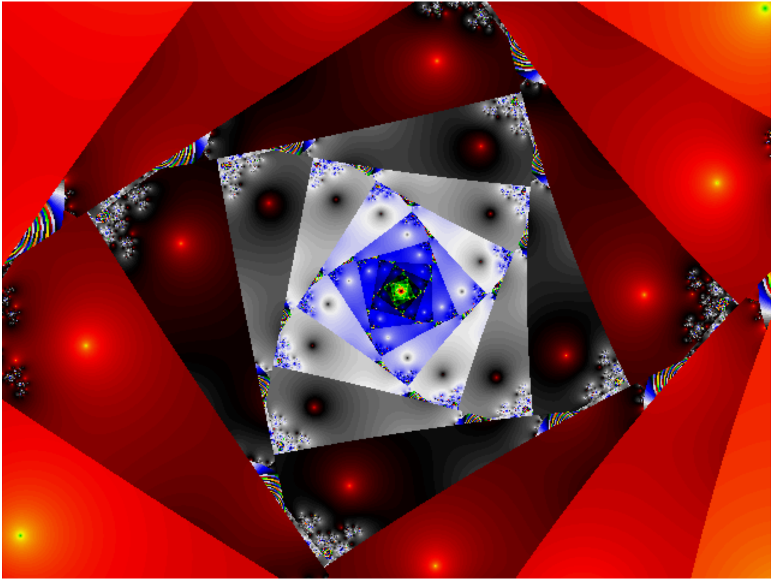
$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Different values of c produce wildly different pictures.

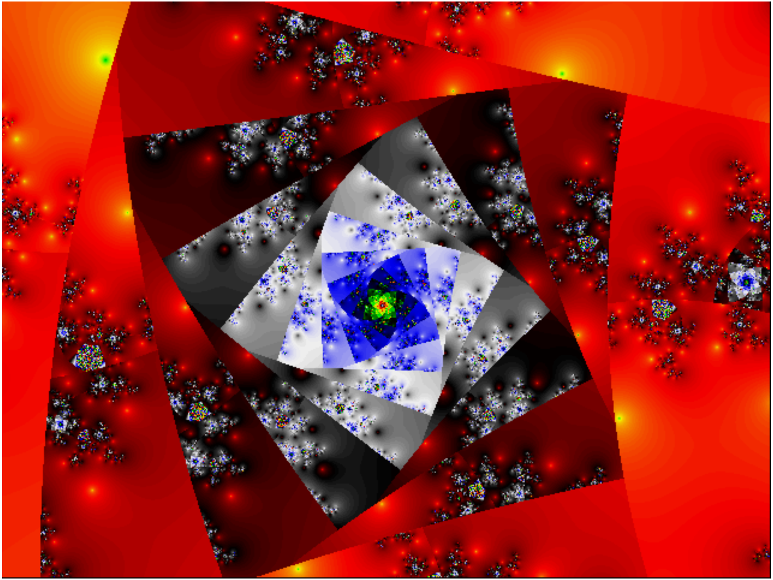
$c \cdot \sin(\ln z)$



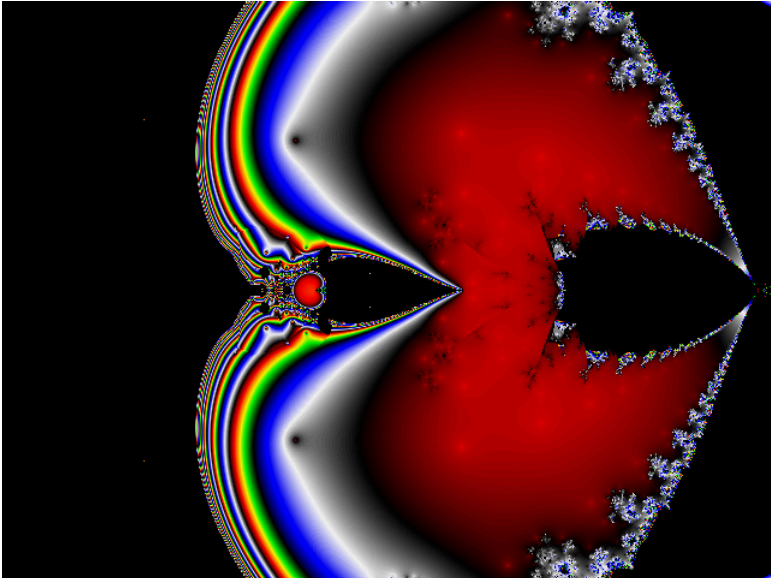
$c \cdot \sin(\ln z)$



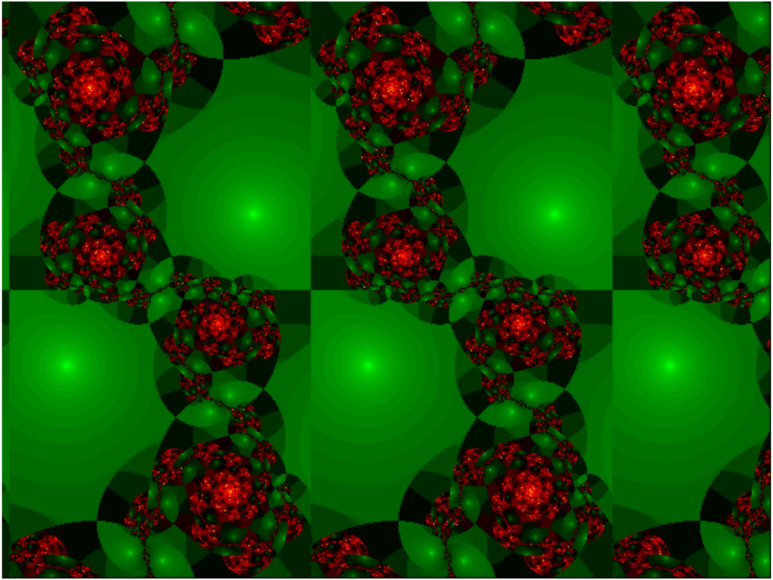
$c \cdot \sin(\ln z)$



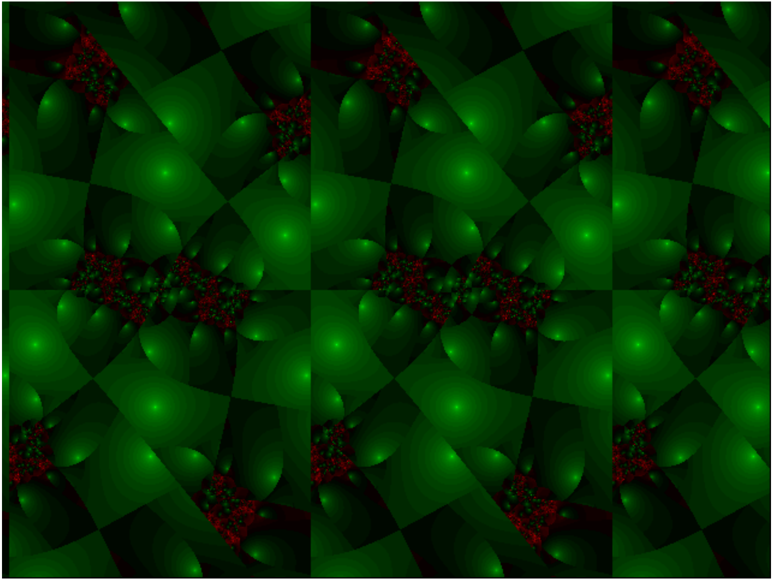
$c \cdot \sin(\ln z)$



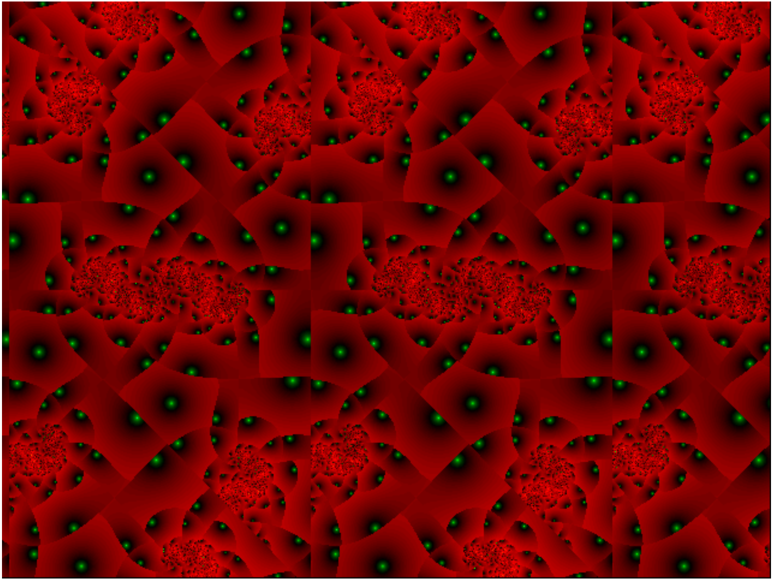
$c \cdot \ln(\sin z)$



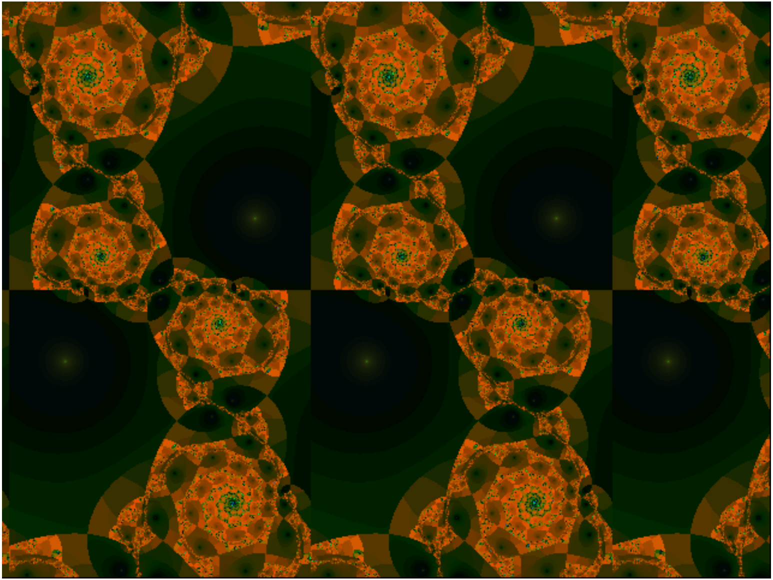
$c \cdot \ln(\sin z)$



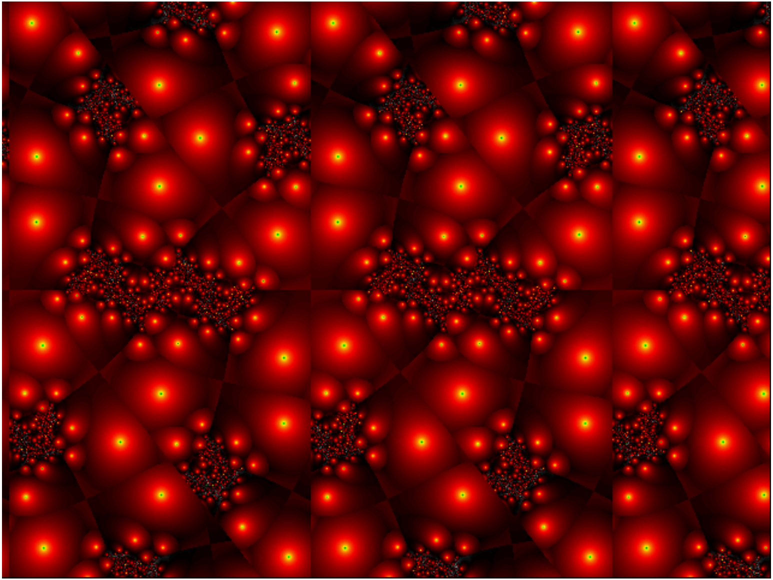
$c \cdot \ln(\sin z)$



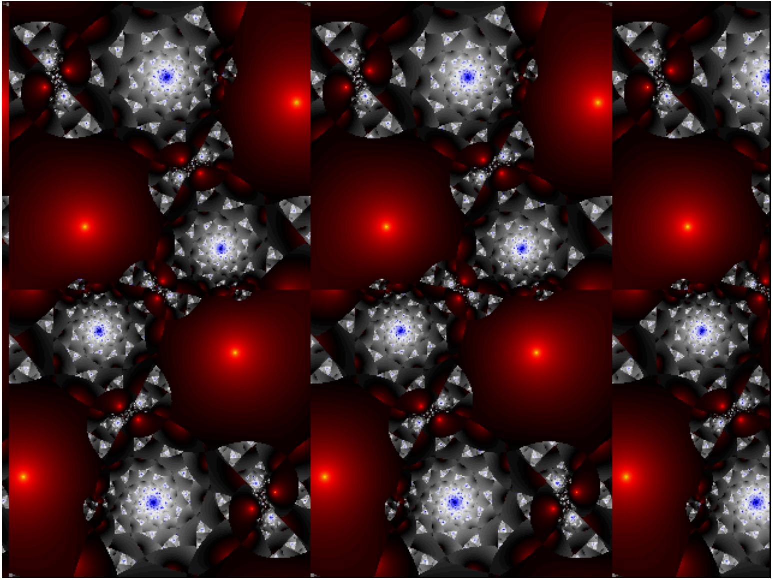
$c \cdot \ln(\sin z)$



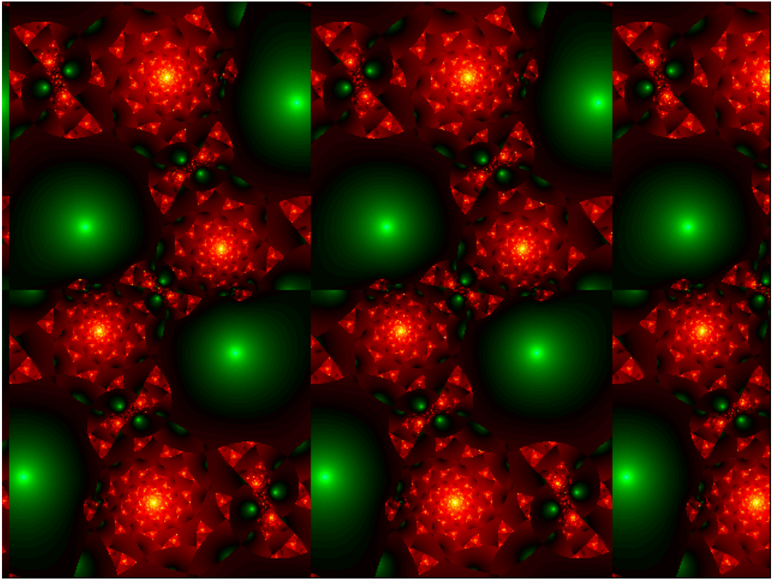
$c \cdot \ln(\sin z)$



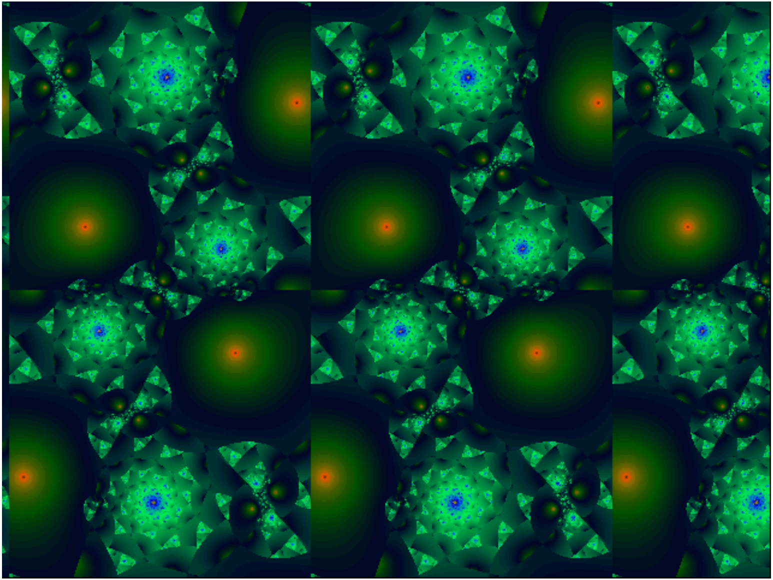
$c \cdot \ln(\sin z)$



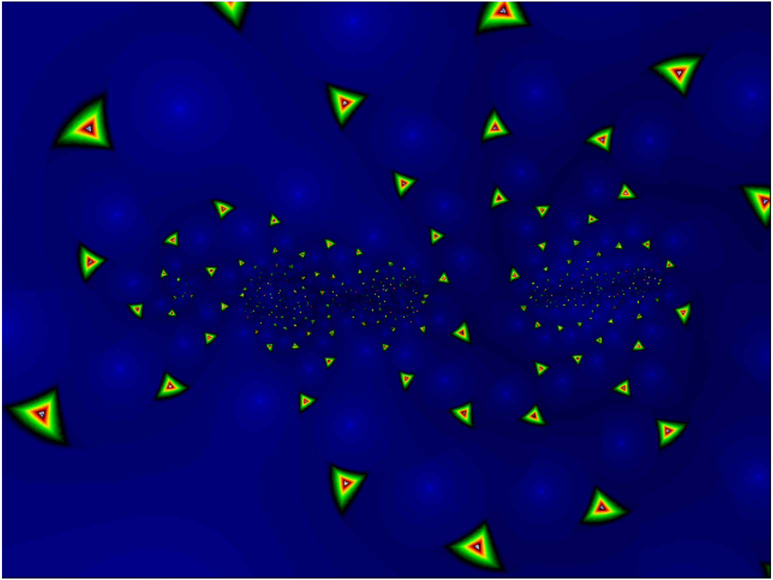
$c \cdot \ln(\sin z)$



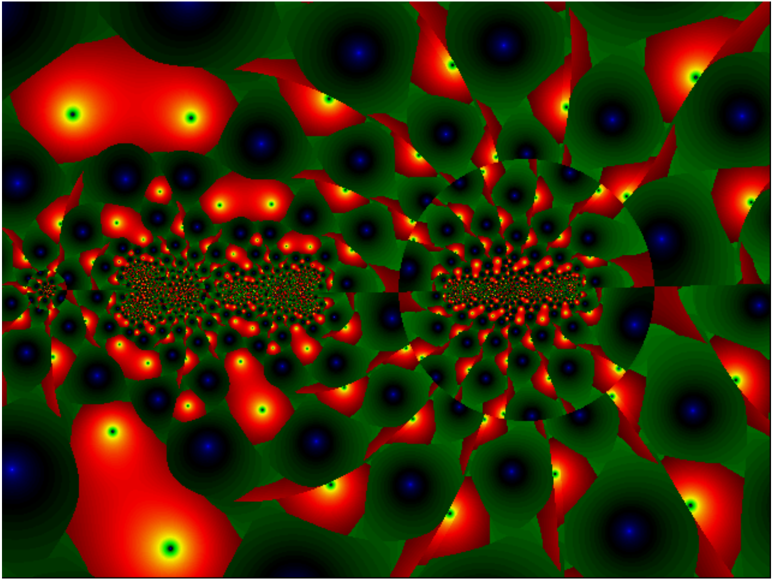
$c \cdot \ln(\sin z)$



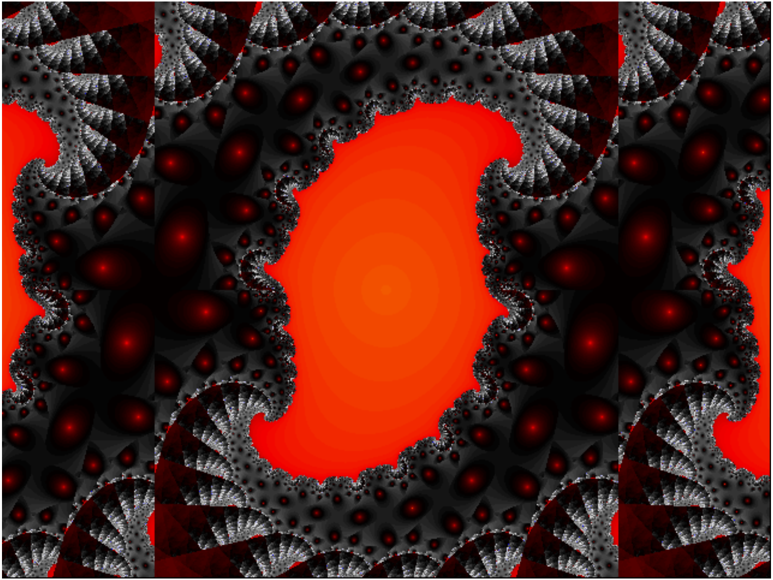
$c \cdot \sin\left(\ln\left(\sin\left(\ln z\right)\right)\right)$



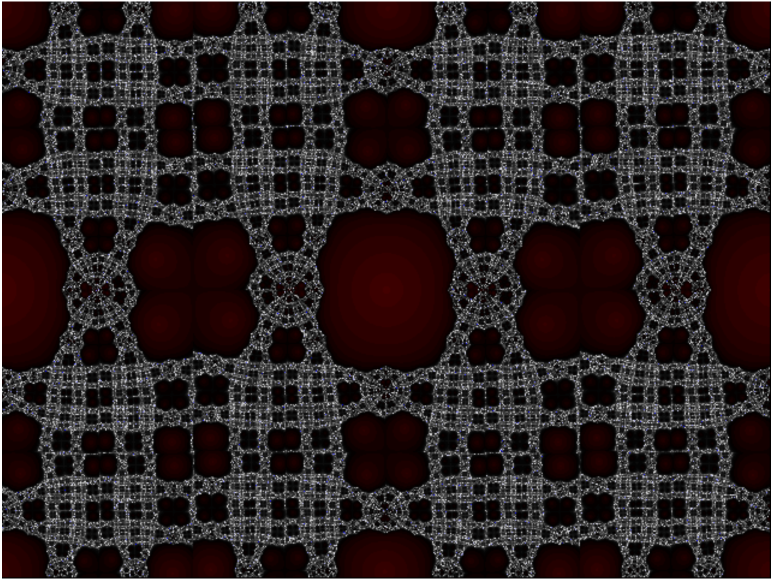
$c \cdot \sin\left(\ln\left(\sin\left(\ln z\right)\right)\right)$



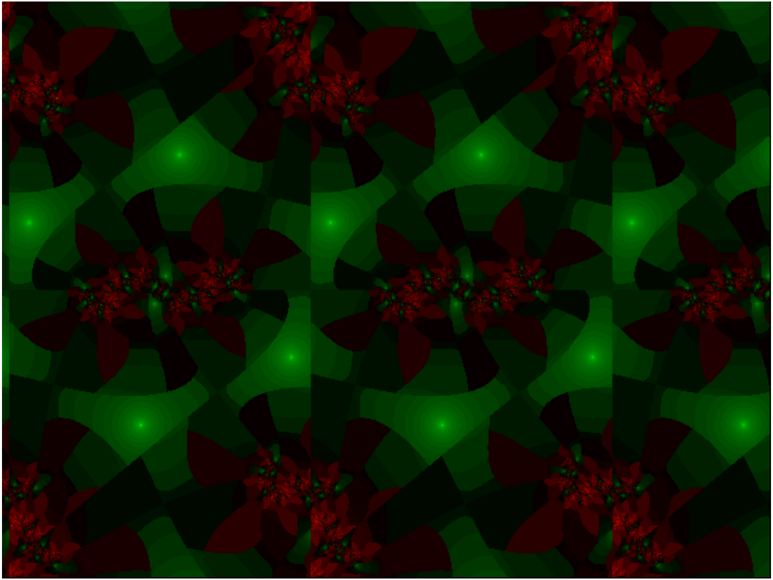
$c \cdot \ln(\cos z)$



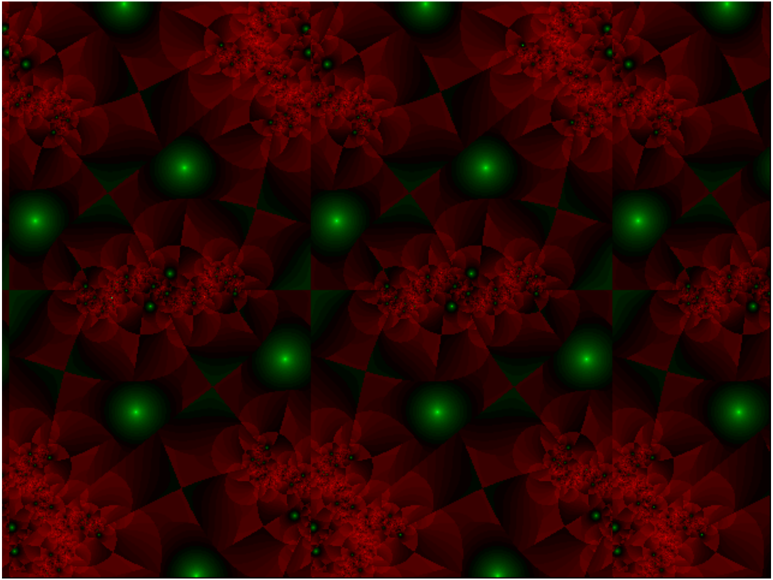
$c \cdot \ln(\cos z)$



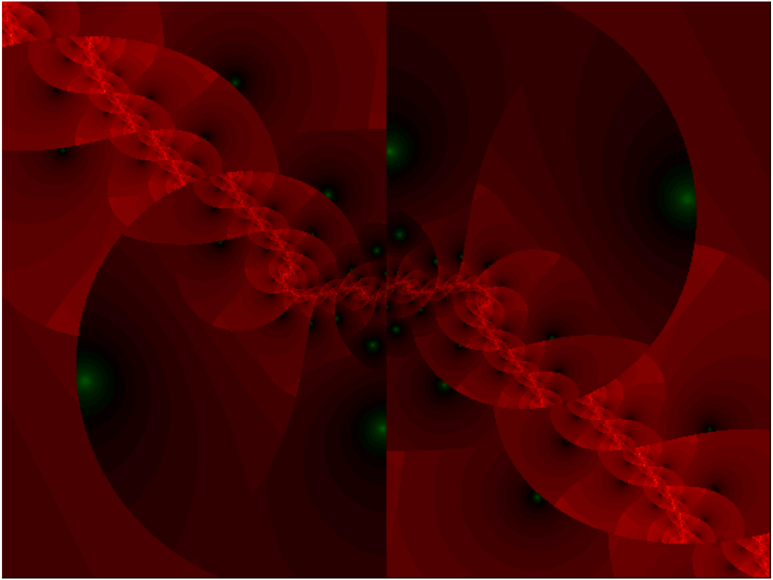
$c \cdot \ln(\csc z)$



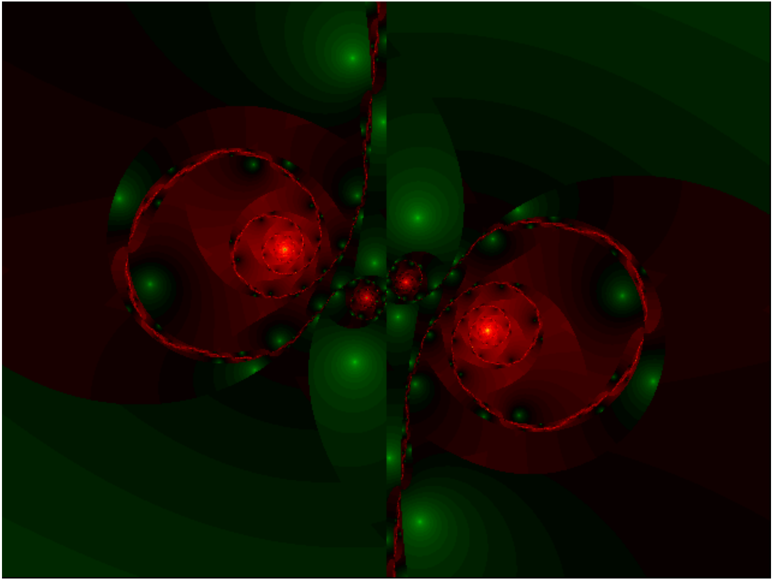
$c \cdot \ln(\csc z)$



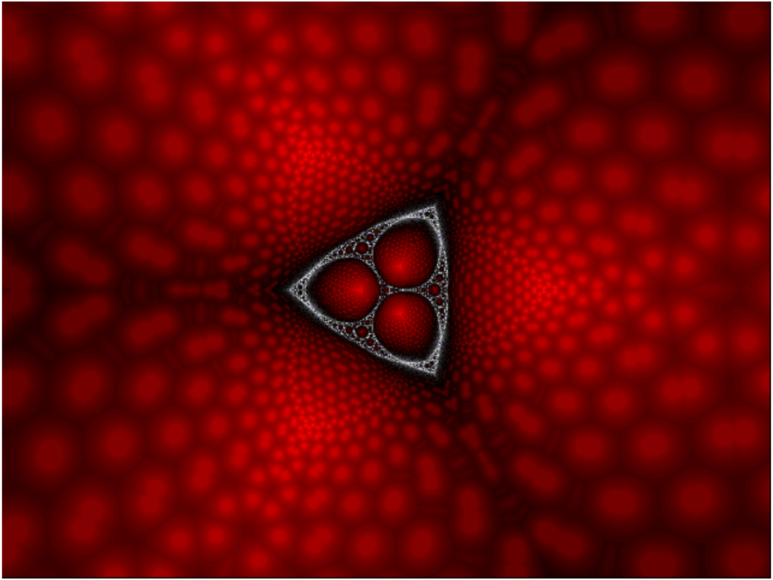
$c \cdot \ln z^4$



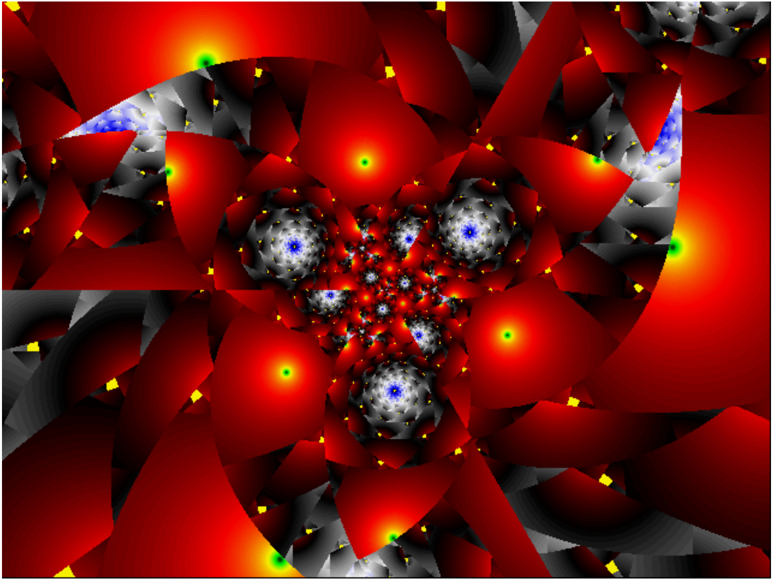
$c \cdot \ln z^2$



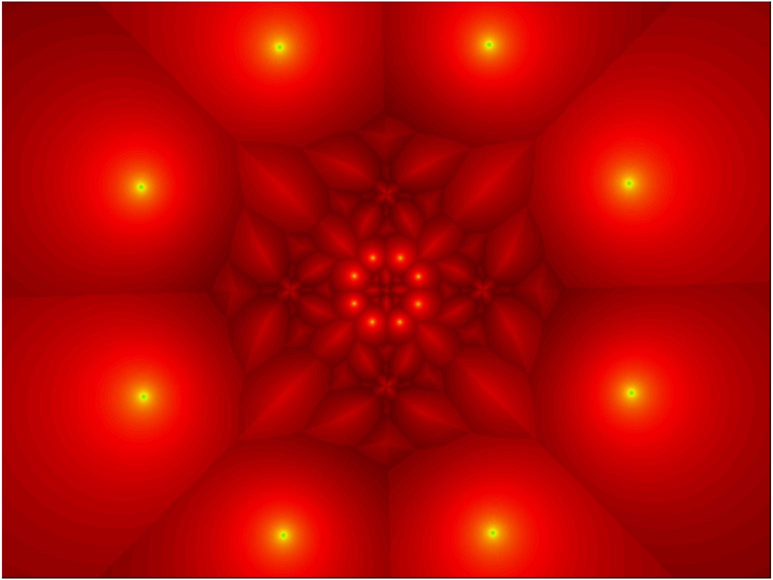
$c \cdot \ln z^3$



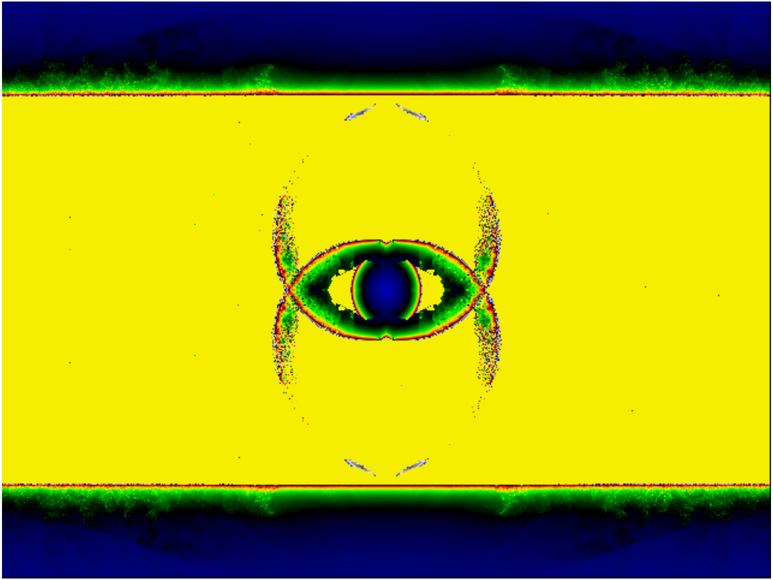
$c \cdot \ln z^3$



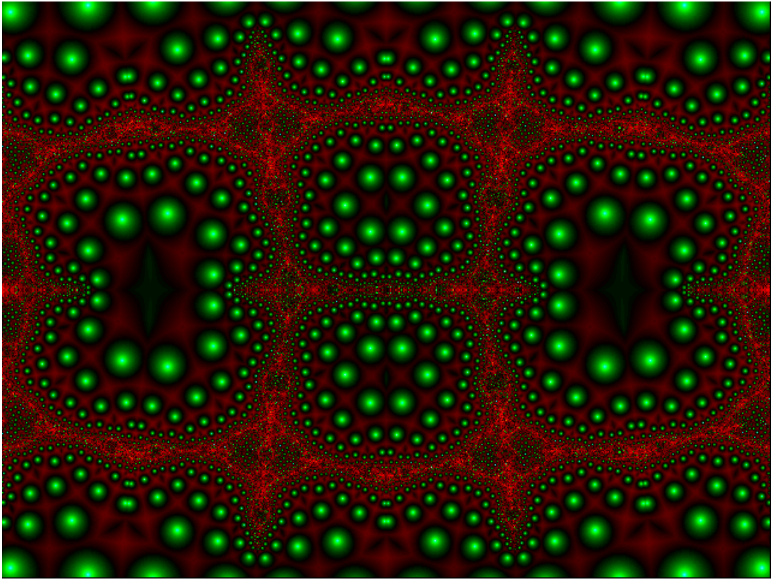
$c \cdot \ln z^4$



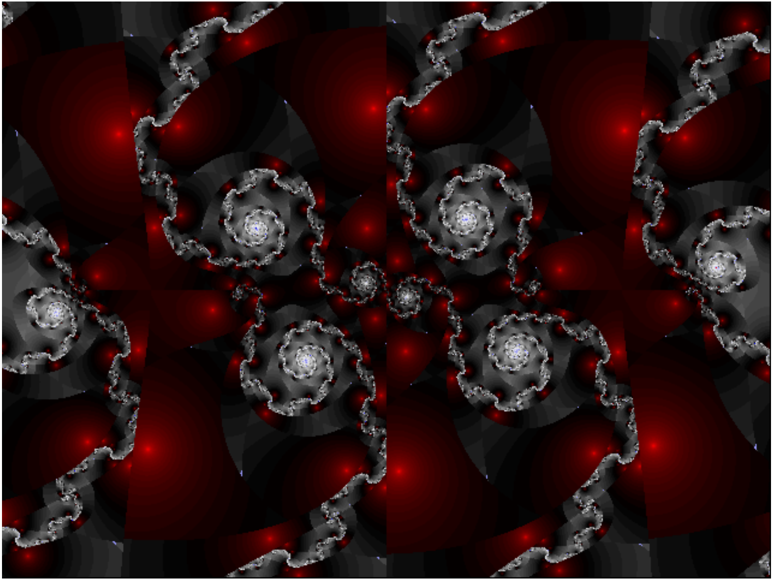
$c \cdot \ln(z \cdot \sin z)$



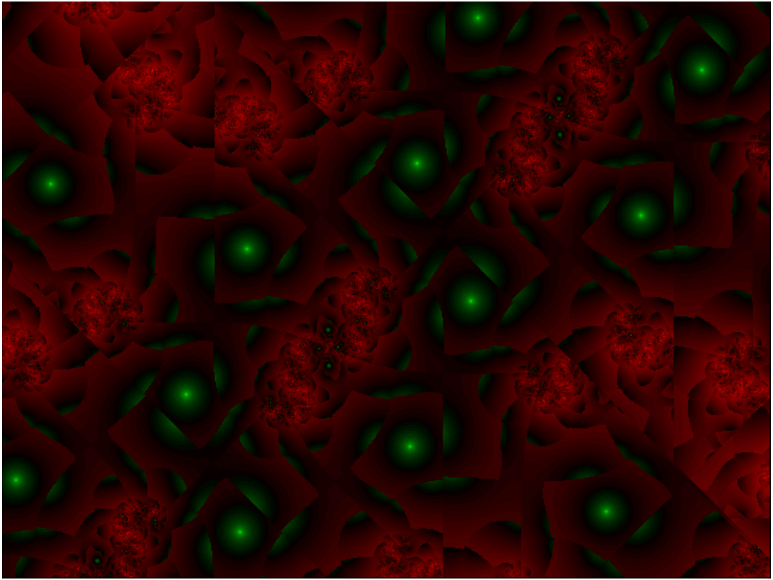
$c \cdot \ln(z \cdot \sin z)$



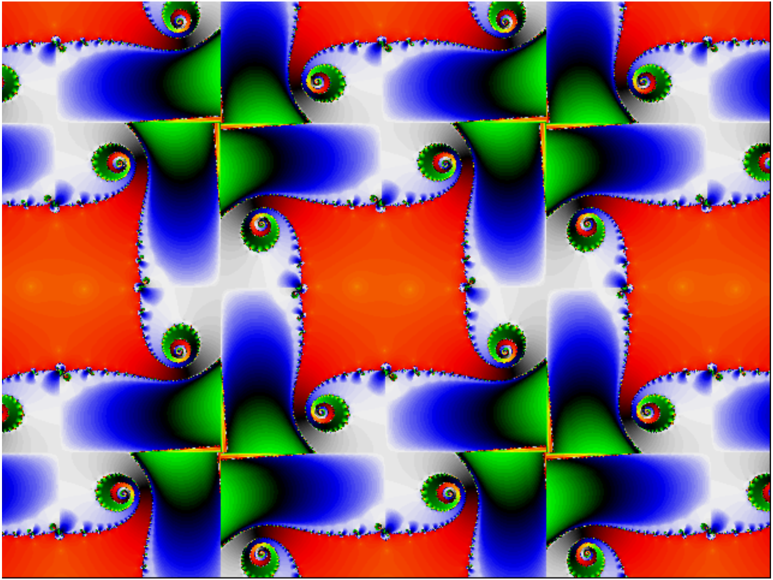
$c \cdot \ln(z \cdot \sin z)$



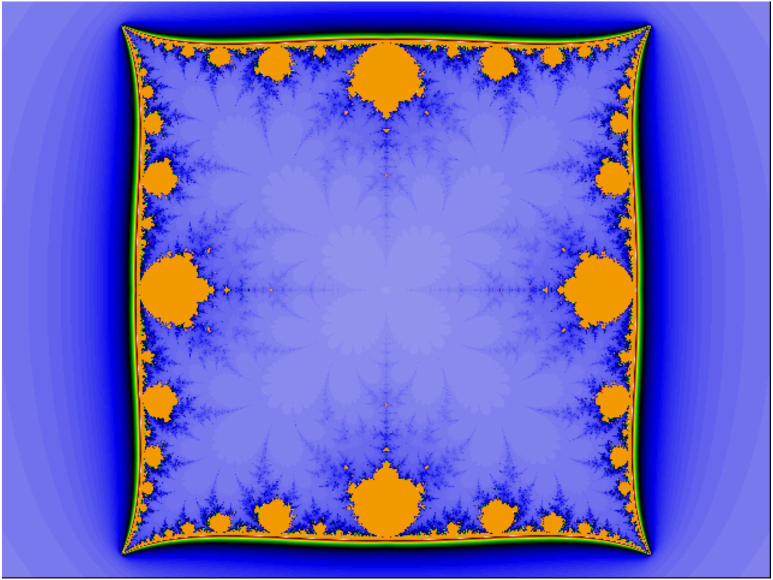
$c \cdot \ln(z \cdot \sin z)$



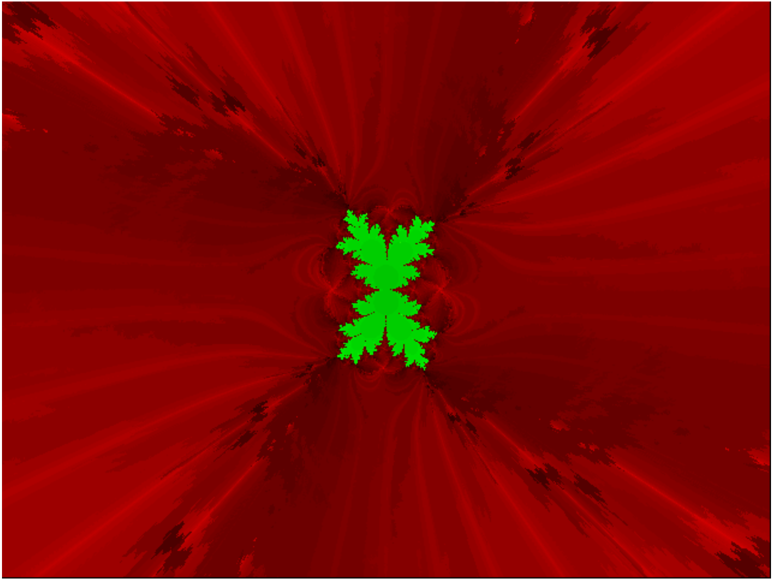
$c \cdot \ln\left(\cos\left(z + c\right)\right)$



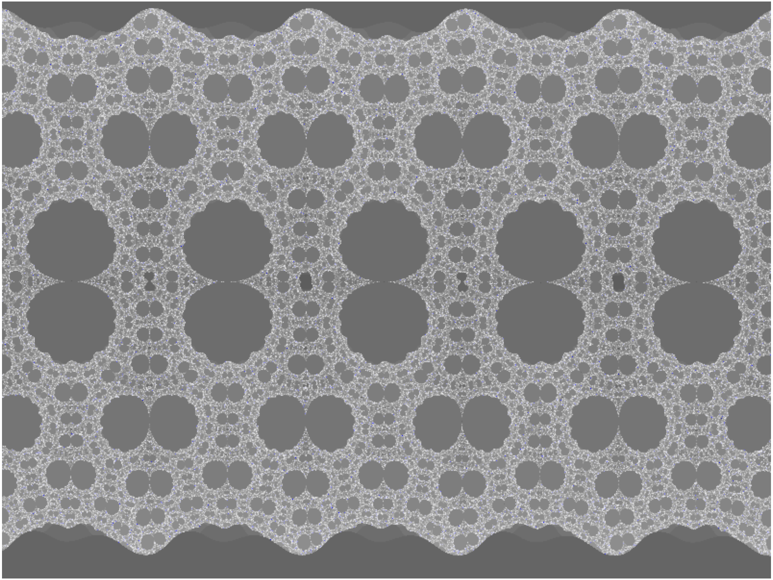
$c \cdot \sec\left(1/z^2\right)$



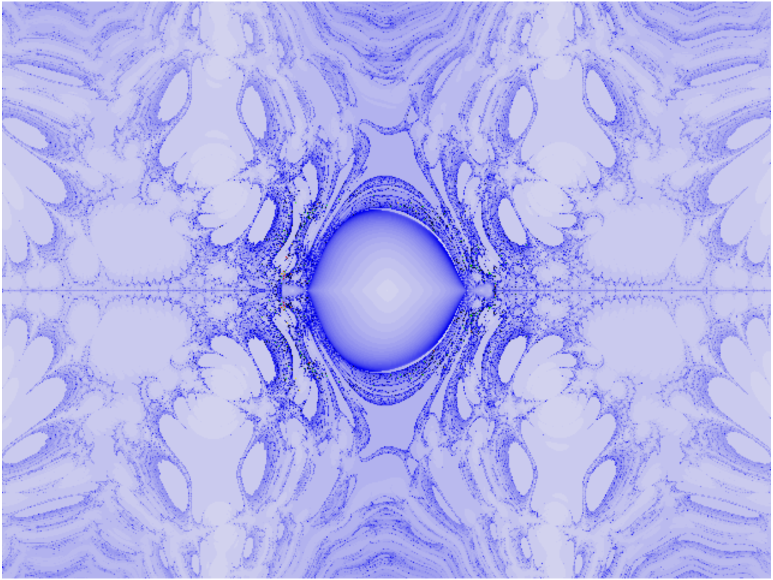
$c \cdot \csc(1/z)$



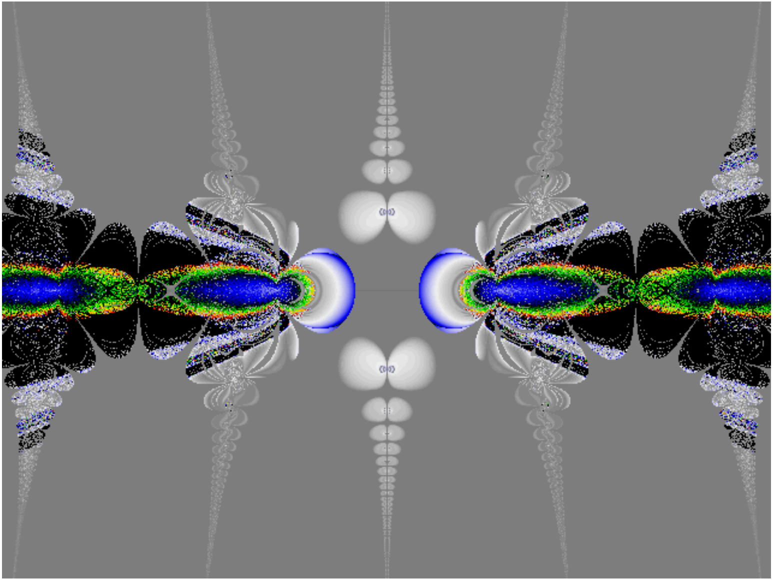
$\sec(cz)$



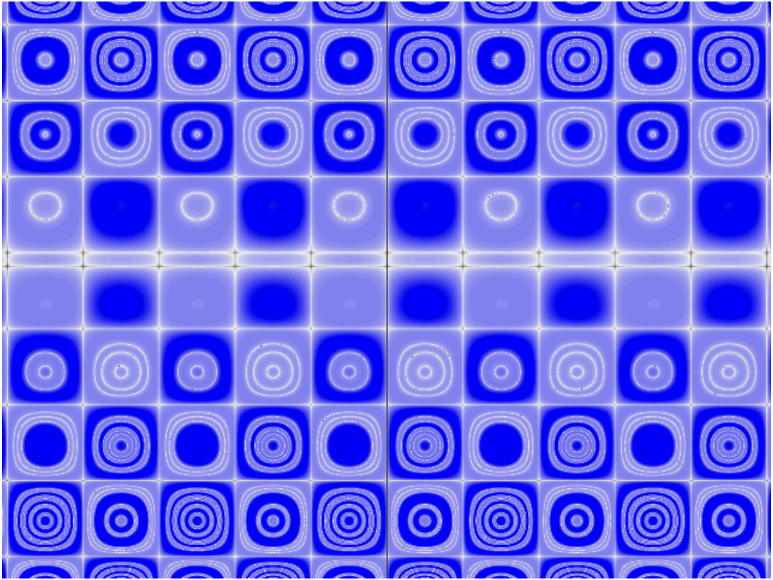
$|z/(\cos(c \cdot \sin z))|$



$\mathrm{Re}(z/(\cos(c \cdot \sin z)))$



$c(1-y)\sin x\cos y$



Raising a complex number to a non-integral power

Exponentiation: $z^p = e^{\ln z^p} = e^{p \ln z}$

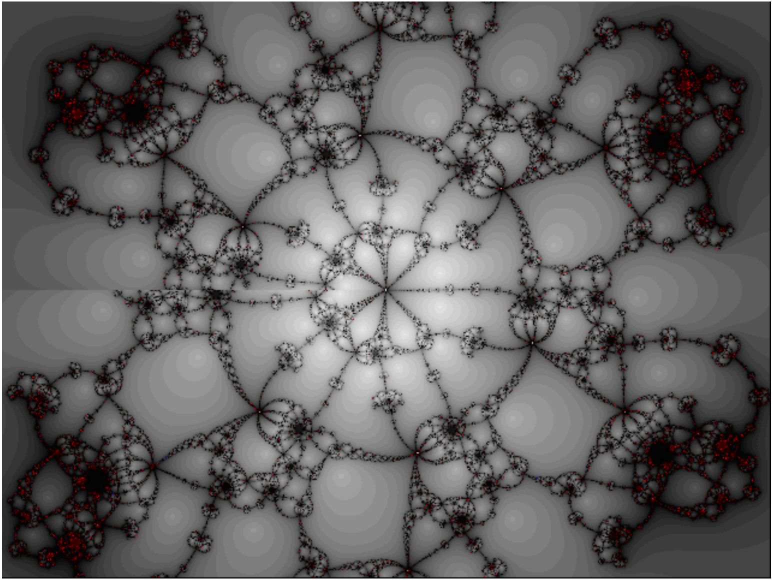
This leads to something interesting:

$i^i = e^{i \ln i} = e^{i(\ln |i| + i \arg i)} = e^{i(\ln 1 + i\pi/2)} = e^{-\pi/2}$

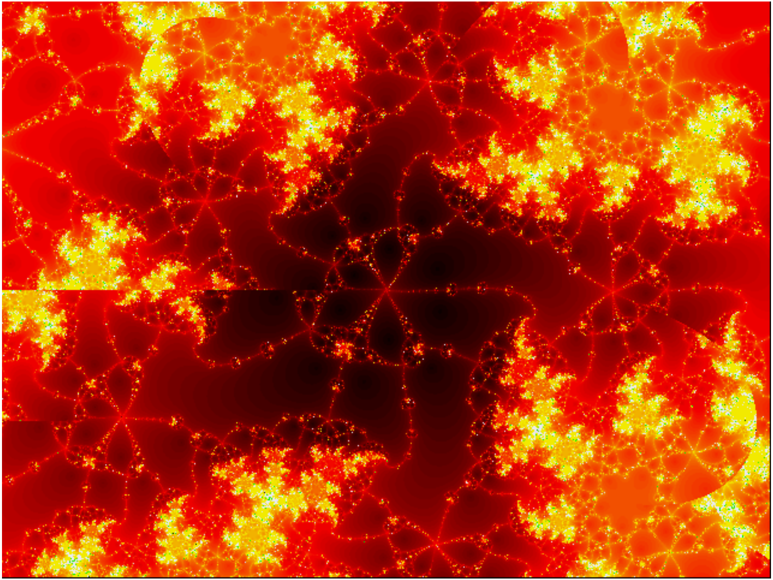
But not without precedent:

$\left(2^{\sqrt{2}}\right)^{\sqrt{2}} = 4$

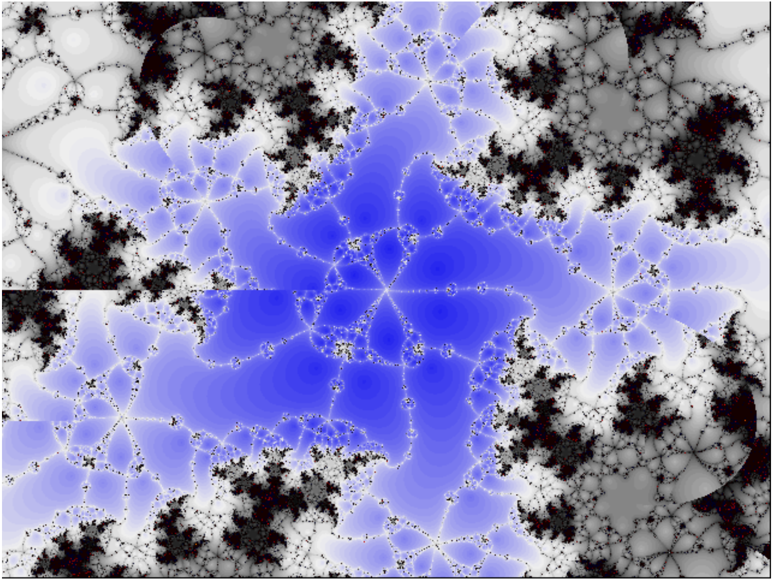
$z - (z^c + z - 1)/(cz^{c-1} + 1)$



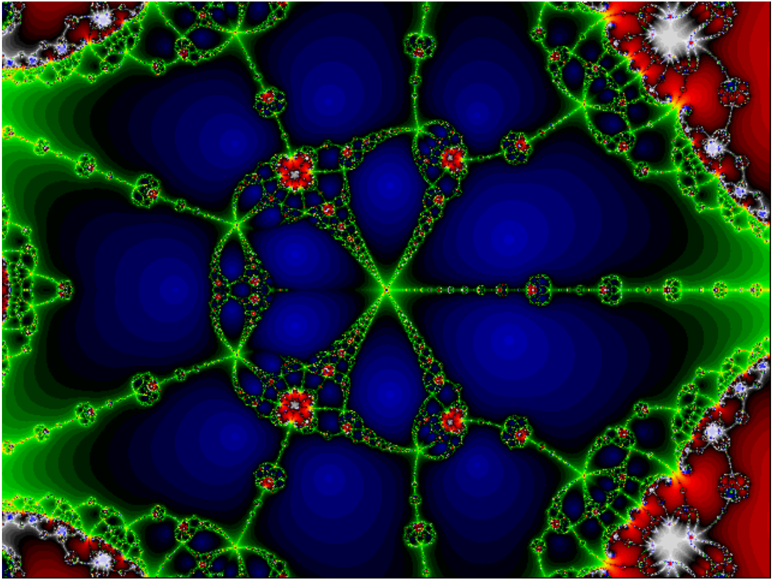
$z-(z^c+z-1)/(cz^{c-1}+1)$



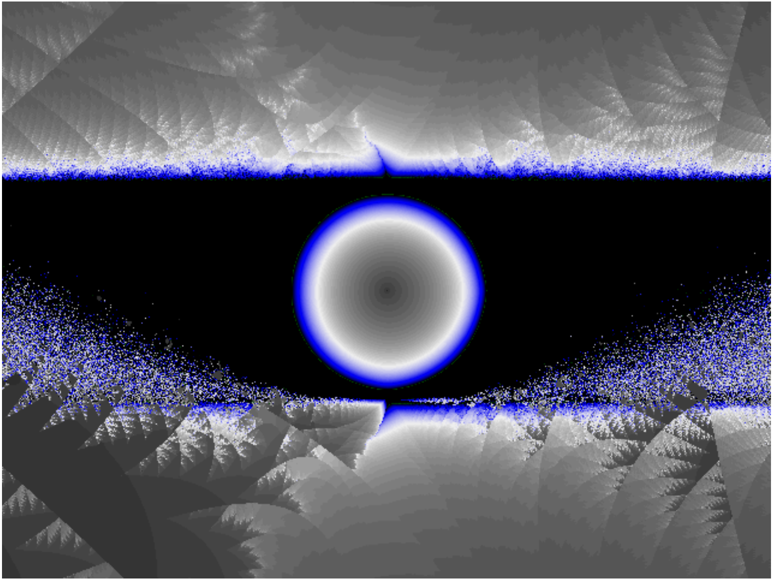
$z - (z^c + z - 1)/(cz^{c-1} + 1)$

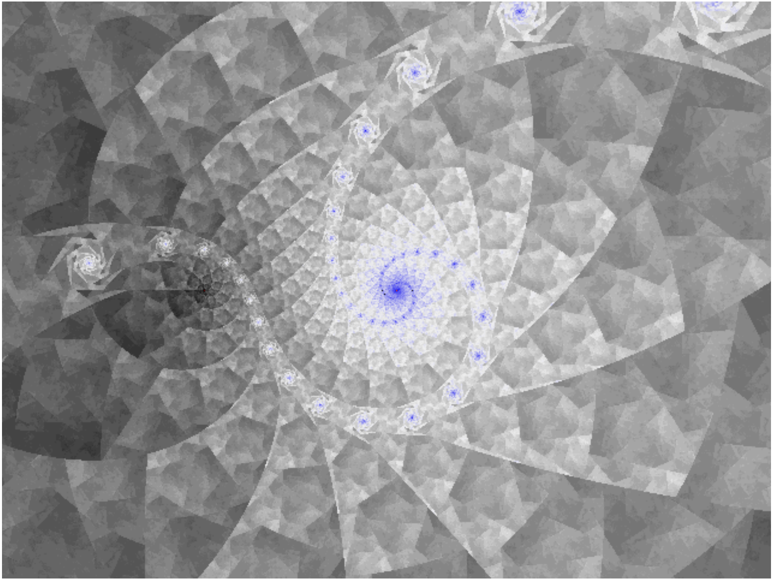


$z - (z^c + z - 1)/(cz^{c-1} + 1)$

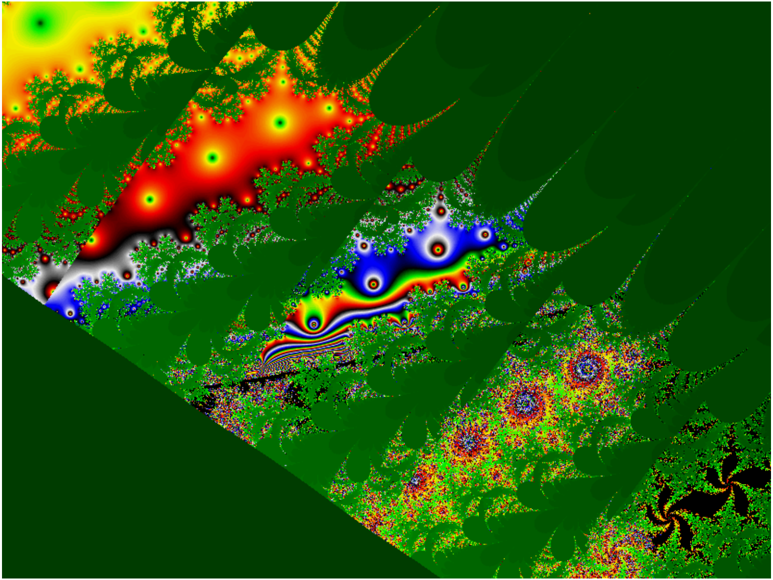


z^{ic}





z^{cz}



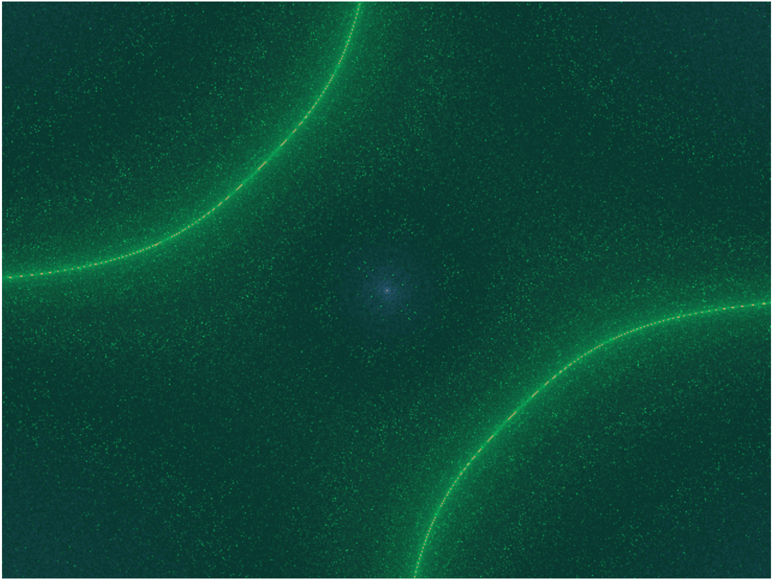
Some unusual functions

"absn" function: $\operatorname{absn}(z) = |z| + i\,\mathrm{Im}(z)$

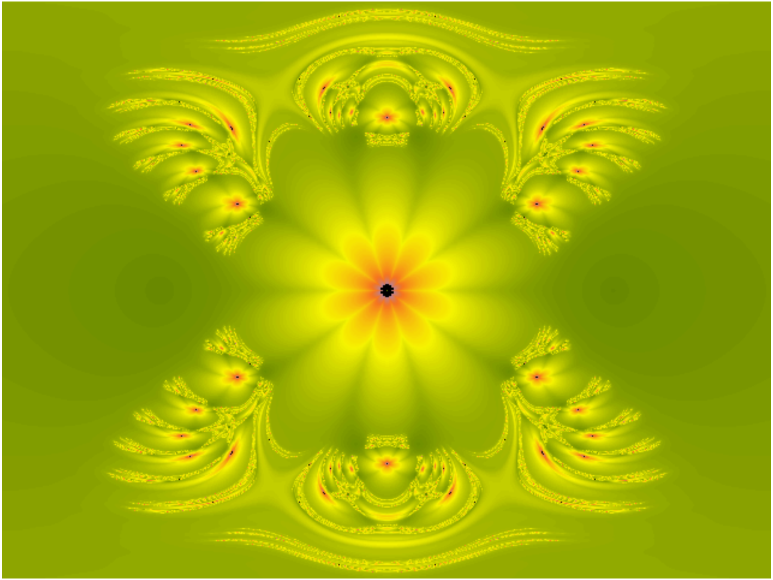
"floor" function: $\operatorname{floor}(x + iy) = \operatorname{floor}(x) + i \cdot \operatorname{floor}(y)$

"and" function: $(x + iy)\&(a + ib) = (x\&a) + i(y\&b)$

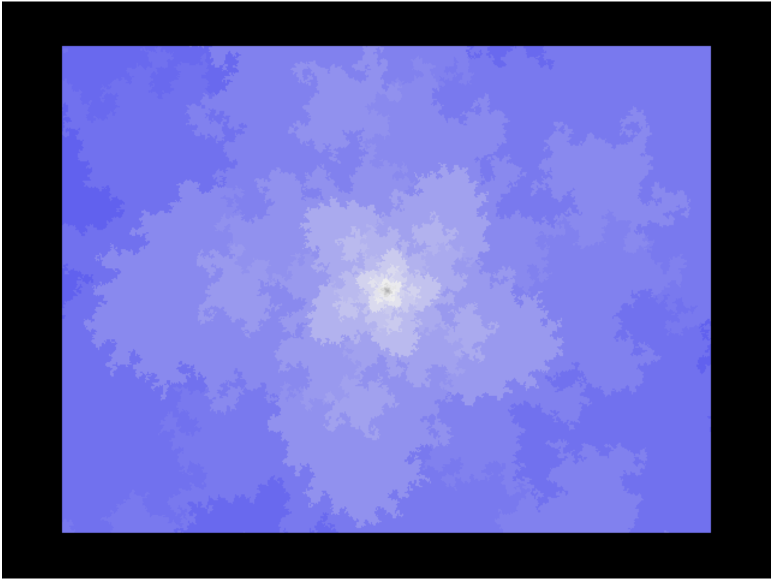
$\text{absn}(z^2) + i \cdot \text{absn}(1/z) + c$



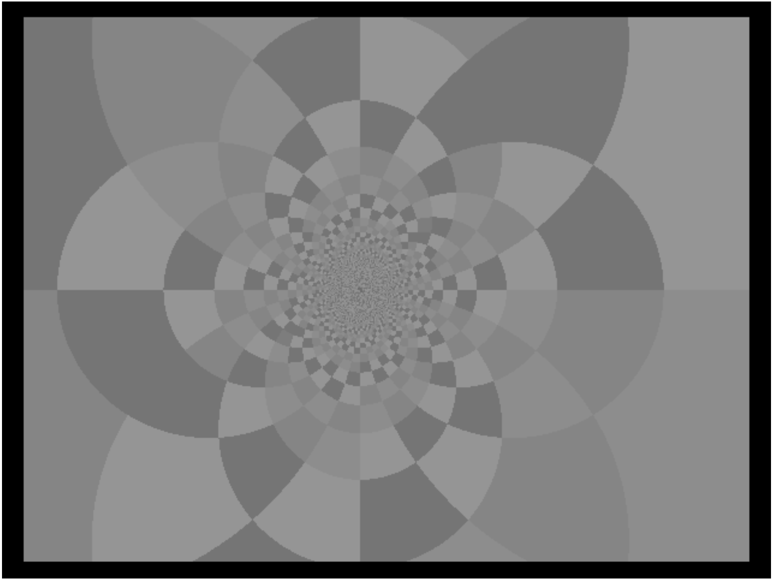
$\operatorname{absn}(z - (z^c - 1)/(cz^{c-1}))$



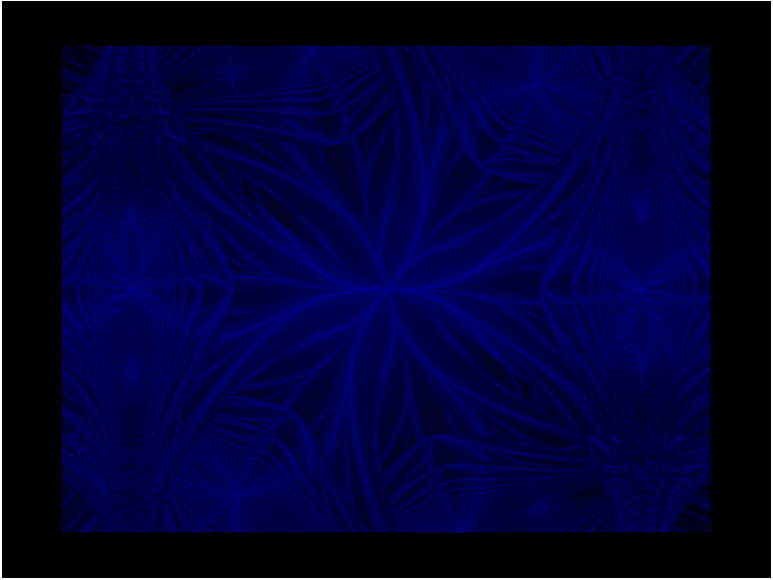
$\text{floor}(cz)$



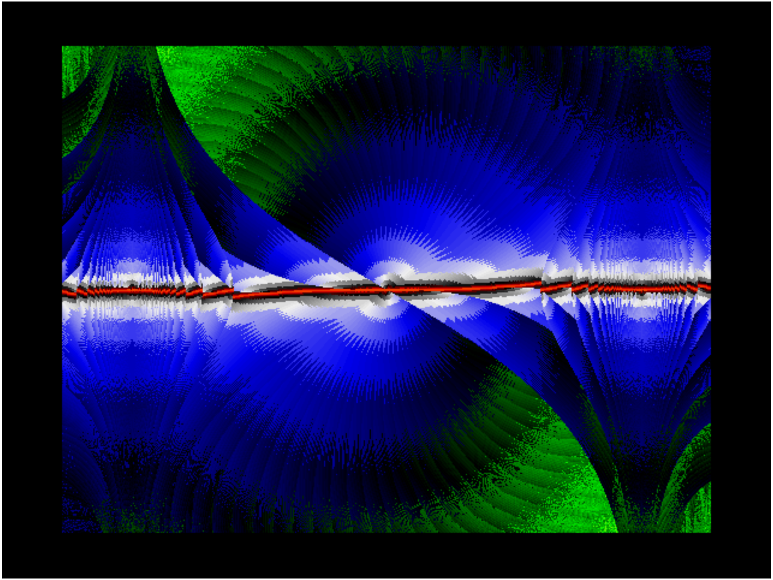
$c\,\text{floor}(\sec(z))$



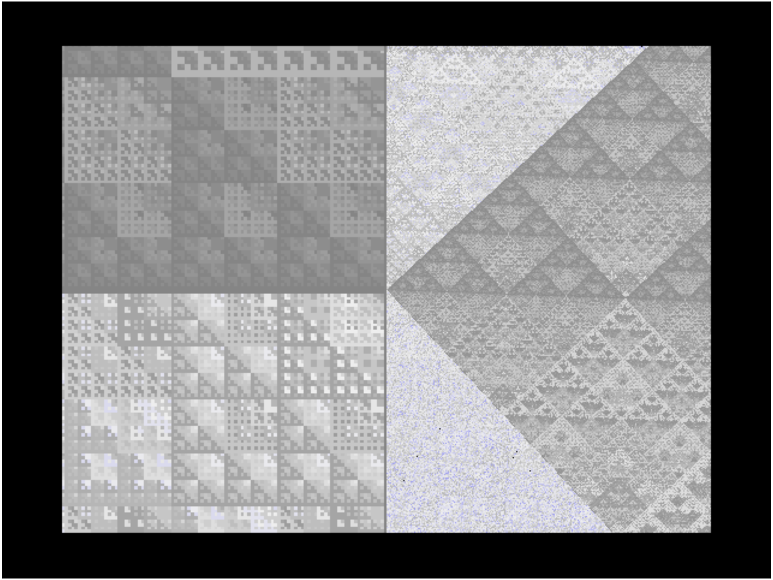
$c(x\% \operatorname{Re}(\sin(z)) + iy\% \operatorname{Im}(\sin(z)))$



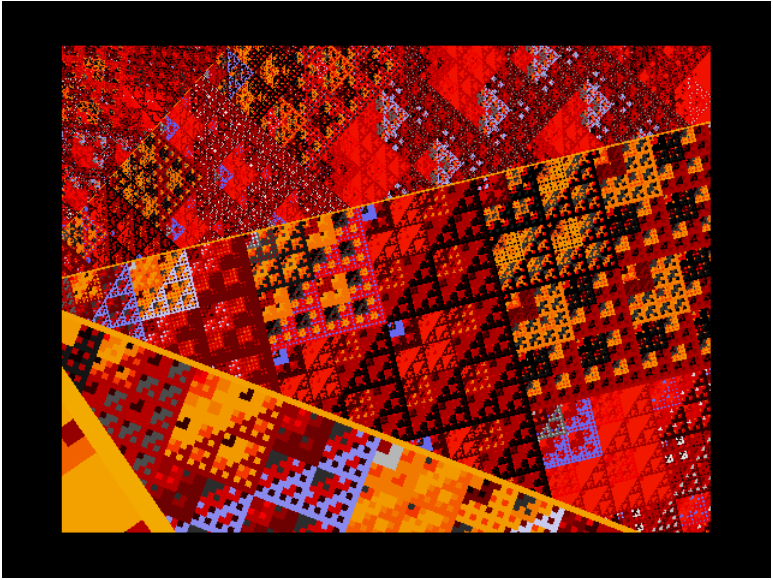
$c(x\%\,\mathrm{Re}(\sin(z)) + iy\%\,\mathrm{Im}(\sin(z)))$



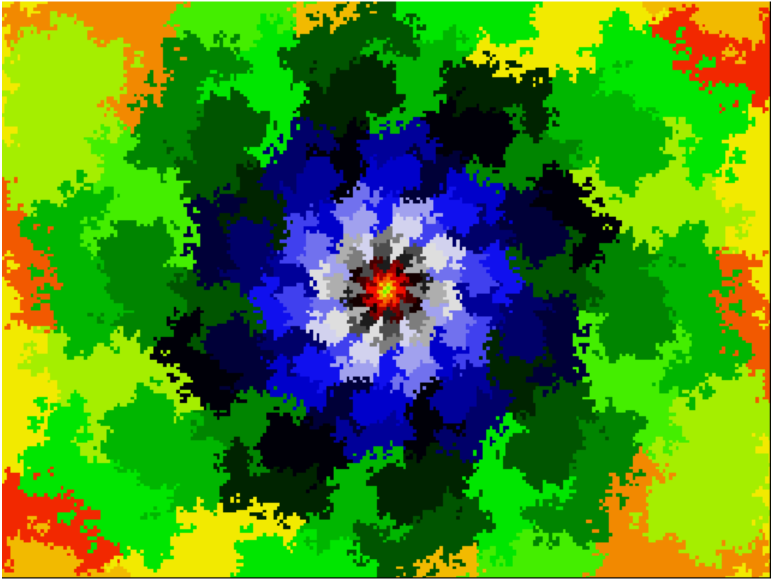
$c((x\&y)\cdot(x<0)+z\cdot(x>0))$



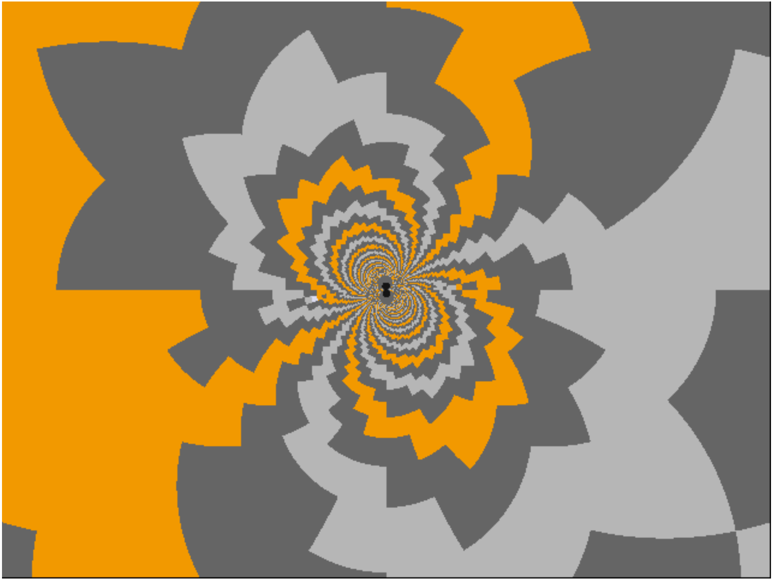
$c((x\&y)\cdot(x<0)+z\cdot(x>0))$



$c(\text{floor}(z) \cdot (x > 0) + \text{ceil}(z) \cdot (x < 0))$

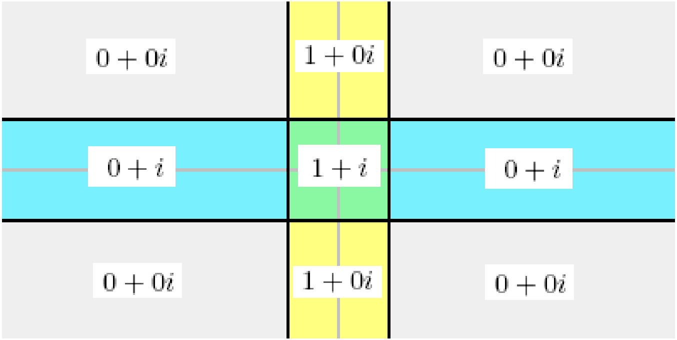


$c \cdot \text{floor}(\csc z \sec z)$

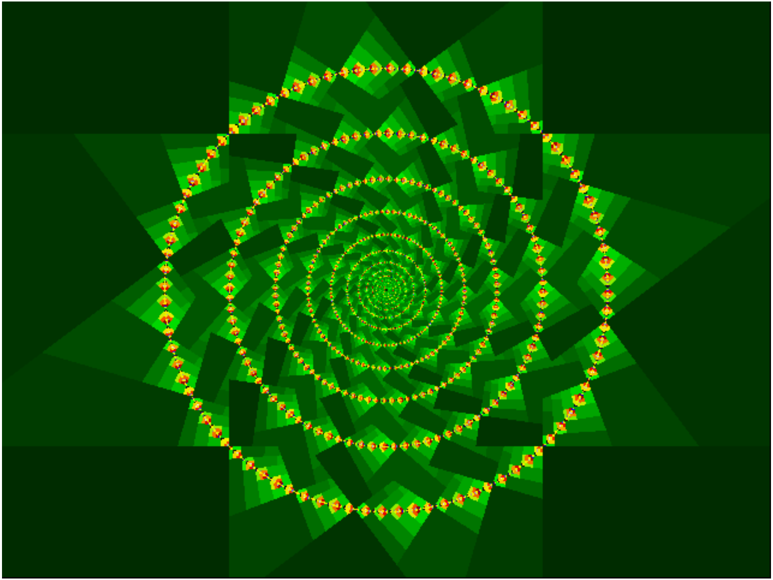


“Not” function

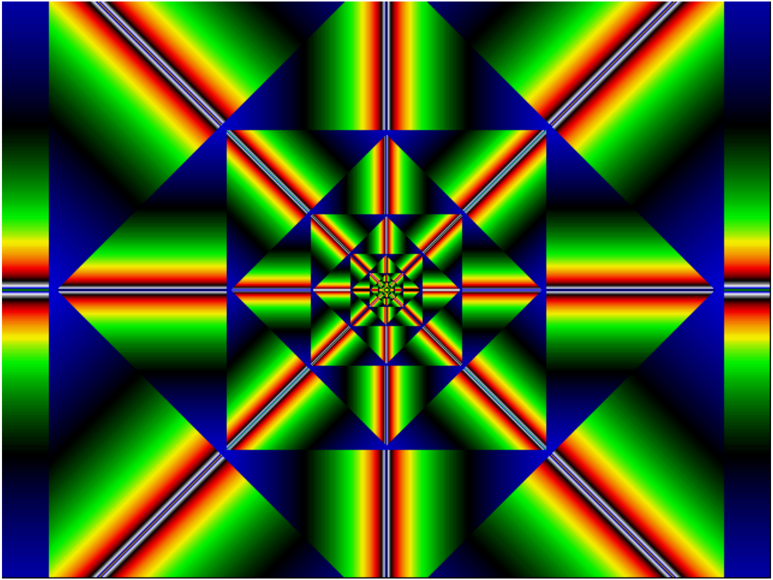
$!(x+iy) = !x + !y$



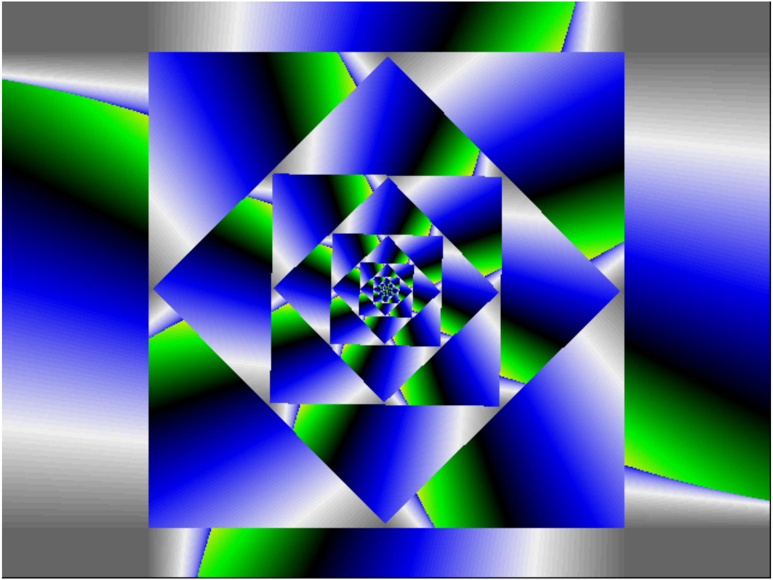
$cz \cdot !z$



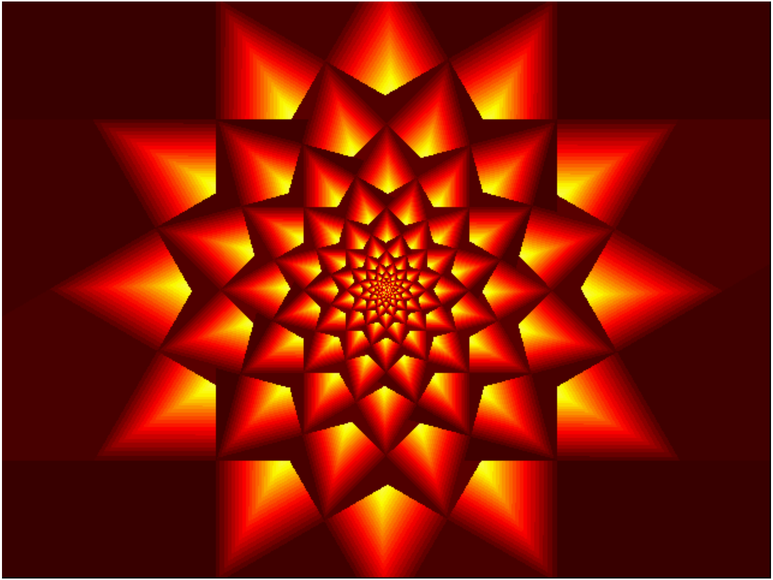
$cz \cdot !z$



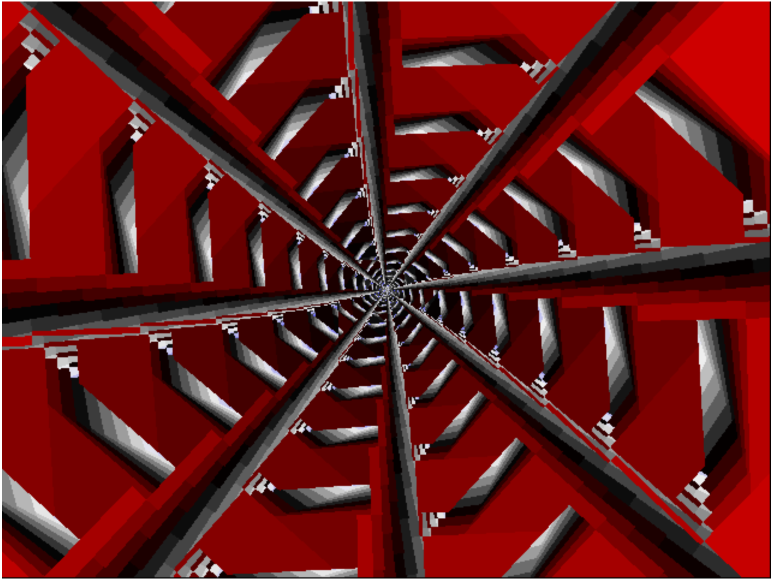
$cz \cdot !z$



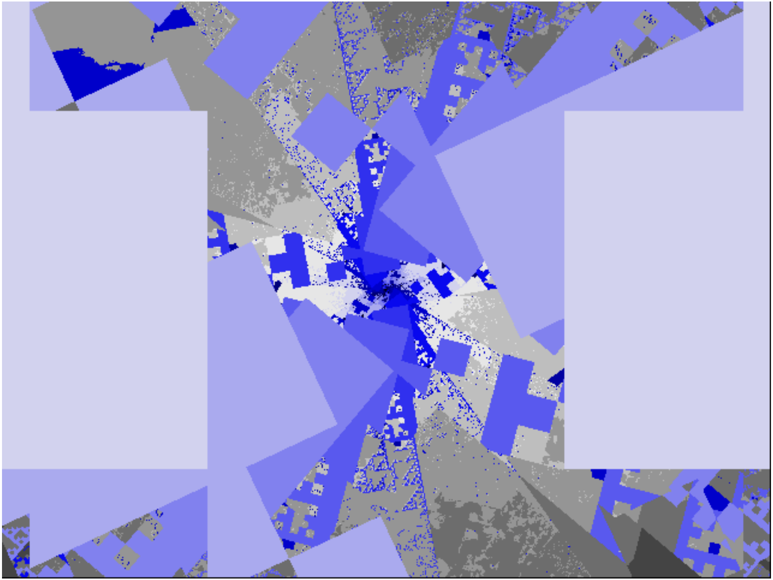
$cz\cdot!z$



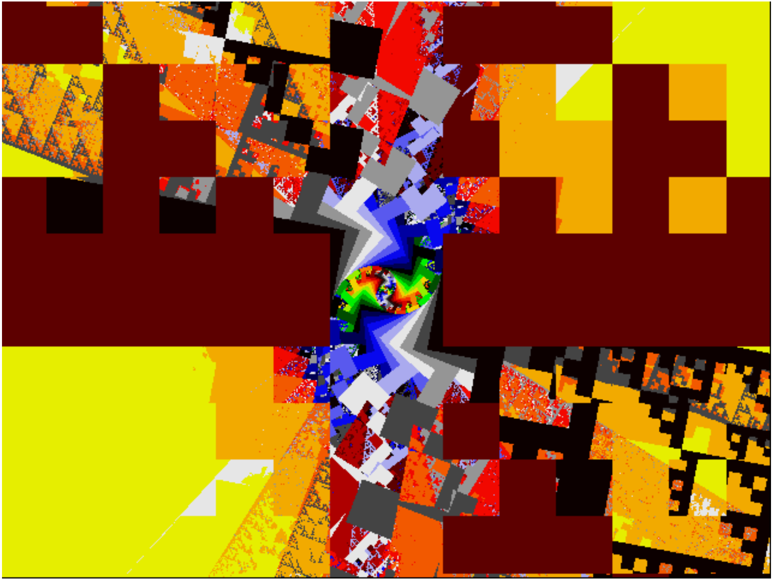
$(cz) \cdot !(z+c)$



$cz(!x + (x\&y))$



$cz(!x + (x\&y))$



$c \cdot \ln\left(\sin\left(z \cdot !z\right)\right)$