Sum List Coloring and Choosability

by

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Abstract

List coloring is a generalization of graph coloring where the vertices of a graph are given lists of colors, and vertices are to be assigned colors from their lists so that adjacent vertices get different colors. Let f be a function assigning list sizes to the vertices of a graph G. The function f is called *choosable* if for every assignment of lists to the vertices of G with list sizes given by f, there exists a coloring of G from the lists (with adjacent vertices receiving different colors). The sum choice number is the minimum over all choosable f of the sum of the list sizes of f. We first answer some questions raised in a paper of Berliner, Bostelmann, Brualdi, and Deatt. Namely, we determine the sum choice number of the Peterson graph, the Cartesian product of paths $P_2 \square P_n$, and the complete bipartite graph $K_{3,n}$. We then determine the sum choice number of $P_3 \square P_n$. Finally, to settle a question of Isaak, Albertson, and Pelsmajer, we investigate the choosability of fan graphs, $P_n \vee K_1$, graphs obtained by joining to a path a single vertex adjacent to each vertex of the path. The techniques developed herein have applications to other classes of graphs.

Chapter 1

Introduction

Graph coloring is one of the oldest and most well-known branches of graph theory. It has a myriad of applications, including register allocation for computer programs, and assignment of frequencies to radio stations. Many problems of both practical and mathematical interest can be modelled as graph coloring problems. Perhaps the most famous example of graph coloring is the Four Color Theorem, which states that any map can be colored with at most four colors in such a way that adjacent regions get different colors. There are numerous variations on graph coloring, and the one we will be interested in here is called list coloring. The vertices of a graph are given lists of colors, and vertices are to be assigned colors from their lists so that adjacent vertices get different colors. List coloring was introduced in the late 1970s, and has been well-studied since. It has applications to a variety of scheduling problems.

1.1 Terminology

We will follow the notation of West [11]. Let G be a graph. We denote the vertex set of G by $V(G)$, and the edge set by $E(G)$. All graphs here are assumed to be finite and simple; loops and multiple edges are of no significance to the coloring problems we consider here. Let $v, w \in V(G)$. We use vw or wv to denote the edge with endpoints v and w, and we say v and w are *adjacent*. A *subgraph* of G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$, and the edges of H have the same endpoints as they have in G. A subgraph H is said to be *induced* if for any pair of vertices $v, w \in V(H)$, $vw \in E(H)$ whenever $vw \in E(G)$. The complement of G, denoted $\overline{G}$, is the graph having the same vertex set as G, and $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. If H is an induced subgraph of G, then $G - H$ denotes the subgraph induced by the vertices of $V(G)\backslash V(H)$. If H consists of a single vertex v, then we will write $G - v$ for this. Let G_1 and G_2 be graphs. The *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *disjoint union* $G_1 + G_2$ is the union of G_1 and G_2 with disjoint vertex sets. The graphs G_1 and G_2 are said to be *isomorphic* if there exists a bijection $g : V(G) \to V(H)$ such that $vw \in E(G_1)$ if and only if $g(v)g(w) \in E(G_2)$.

A *neighbor* of a given vertex is a vertex to which it is adjacent. The *neighborhood* of a vertex v, denoted $N(v)$, is the subset of $V(G)$ consisting of all the vertices adjacent to v. If H is a subgraph of G, we write $N_H(v)$ to denote the set consisting of all the neighbors of v which are in H. The *degree* of a vertex v, denoted $\deg(v)$, is the size of the neighborhood of v. If H is a subgraph of G, $\deg_H(v)$ denotes the number of vertices of H to which v is adjacent. A *leaf* in a graph is a vertex of degree 1. An *independent set* in a graph is a subset of pairwise non-adjacent vertices. A vertex is called *isolated* if it is adjacent to no other vertices.

The *complete graph* on n vertices, denoted K_n, is the graph on n vertices in which every vertex is adjacent to every other vertex. A graph G is called *bipartite* if $V(G) = V(X) \cup V(Y)$, where X and Y are independent sets. The *complete bipartite graph* with partite sets of size p and q, denoted $K_{p,q}$, has all possible edges between the partite sets (see Figure 1.2a). A *path* on n vertices is a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_iv_{i+1} : i = 1, \ldots, n-1\}$. A *cycle* on n vertices is a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_iv_{i+1} : i = 1, \ldots, n\}$, where the addition is modulo n. The *length* of a path or cycle is the number of vertices. The graph which is a path on n vertices is denoted by P_n, and similarly the cycle on n vertices is denoted C_n. An *odd (even) cycle* is a cycle with odd (even) length. A graph is called *connected* if any two vertices lie on a path. A *tree* is a connected graph with no cycles. A graph is called *2-connected* if upon the deletion of any vertex, the resulting graph is still connected. A *block* of a graph is a maximal 2-connected subgraph. For example, in Figure 1.1, the blocks are the subgraphs induced by $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_{11}, v_{12}\}$, $\{v_4, v_5, v_{10}, v_{11}\}$, and $\{v_5, v_6, v_7, v_8, v_9\}$.

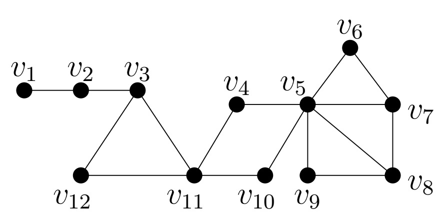


Figure 1.1: Example of blocks

A *planar graph* is a graph which can be embedded in the plane, that is, it can be drawn so that its edges never cross. An *outerplanar graph* is a planar graph such that all the vertices lie on a circle in the plane, and any edges are on or within the circle. The *Peterson graph* is the graph in which the vertices correspond to the 2-element subsets of a 5-element set, with an edge between two vertices if and only if their corresponding 2-element subsets are disjoint (see Figure 1.2b). The *Cartesian product* of two graphs G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$, with an edge between (v, w) and (v', w') if and only if $w = w'$ and $vv' \in E(G_1)$, or $v = v'$ and $ww' \in E(G_2)$. The *join* of two graphs G_1

and G_2, denoted $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$, and edge set $E(G_1) \cup E(G_2) \cup \{v_1 v_2 : v_1 \in E(G_1), v_2 \in E(G_2)\}$. For example, the complete bipartite graph $K_{p,q}$ can be expressed as $\overline{K}_p \vee \overline{K}_q$. The *line graph* $L(G)$ of a graph G is obtained from G by letting $V(L(G)) = E(G)$ and assigning an edge between vertices if and only if their corresponding edges in G share an endpoint. A *theta graph*, θ_{k_1,k_2,k_3}, is a graph consisting of two vertices connected by three internally vertex disjoint paths, having k_1, k_2, and k_3 internal vertices, respectively, $0 \leq k_1 \leq k_2 \leq k_3$ (see Figure 1.2c). A *fan graph* F_n is the graph $P_n \vee K_1$, obtained by joining a vertex to a path.

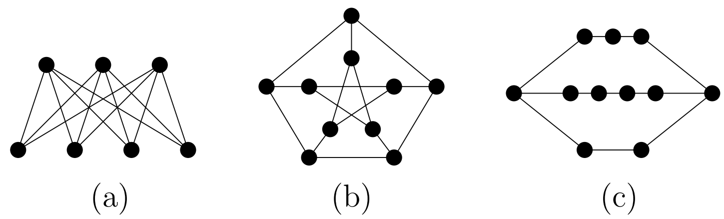


Figure 1.2: $K_{3,4}$, the Peterson graph, and $\theta_{3,4,2}$

1.2 Graph coloring

A *coloring* of a graph G is an assignment of labels from a set C to the vertices of G. Formally, it is a map $c : V(G) \to \mathrm{C}$. The labels are usually called colors, as the concept of graph coloring arose from a map coloring problem. We will assume that the elements of C are positive integers. A *proper coloring* is a coloring such that adjacent vertices are assigned different colors. We say G is *k-colorable* if there exists a proper coloring c of G such that $|\{c(v) : v \in V(G)\}| = k$. The *chromatic number* of a graph, denoted $\chi(G)$, is the minimum integer k such that G is k-colorable. In other words, it is the least number of colors needed to properly color the graph.

Greedy coloring easily shows $\chi(G) \leq \Delta(G)+1$, where $\Delta(G)$ denotes the maximum vertex degree of G. By greedy coloring we mean that we choose some ordering of the vertices, and color the vertices in order such that each is assigned the least color (taking the colors to be positive integers) not assigned to any adjacent vertex of smaller index. One can improve the upper bound slightly to $\chi(G) \leq \Delta(G)$, provided G is a connected graph which is neither an odd cycle nor a complete graph. This is Brooks' Theorem. An obvious lower bound for $\chi(G)$ is $\omega(G)$, the *clique number* of G, which is the size of the largest subset of $V(G)$ whose elements are pairwise adjacent. One may easily prove that $\chi(G) = 2$ if and only if G has no odd cycles. However, if $k \geq 3$, determining if a graph is k-colorable is NP-complete (see [11], for example).

1.3 List coloring

List coloring is a generalization of graph coloring in which each vertex is given a list of permissible colors, and one tries to assign colors to vertices such that each vertex is assigned a color from its list, with adjacent vertices getting different colors. More formally, a *size function* $f : V \to \mathbb{Z}$ assigns to each vertex a list size. An *f-assignment* $\mathcal{C} : V \to 2^{\mathrm{C}}$ is an assignment of lists of colors to each vertex v such that $|\mathcal{C}(v)| = f(v)$. A *$\mathcal{C}$-coloring* is a function $c : V \to \mathrm{C}$ such that $c(v) \in \mathcal{C}(v)$, and c is called *proper* if adjacent vertices get different colors. If G has a proper $\mathcal{C}$-coloring we say G is *$\mathcal{C}$-colorable*, or simply that $\mathcal{C}$ is *colorable*. A size function f is called *choosable* if every f-assignment has a proper coloring, and if we wish to emphasize the graph, we may instead say that (G, f) is choosable. A choosable size function is called a *choice function*. By convention, if $f(v) \leq 0$ for some v, then any f-assignment $\mathcal{C}$ has $\mathcal{C}(v) = \emptyset$, and f is not choosable. Lists such as $\{1, 2, 3\}$ are written in the abbreviated form 123.

As an example, consider $(P_2 \,\square\, P_3, f)$ with $f \equiv 2$, and let $\mathcal{C}$ be the f-assignment shown in Figure 1.3. Any proper $\mathcal{C}$-coloring must either use color 1 on the top middle

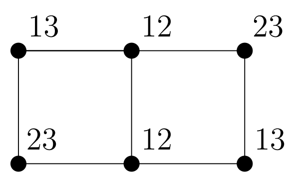


Figure 1.3: A list coloring example

vertex and color 2 on the bottom middle vertex or vice-versa. In the former case such a coloring would be forced to use color 3 on both vertices on the left, and in the latter case color 3 is forced to be used on both vertices on the right. Hence there is no proper coloring from $\mathcal{C}$. This shows that $f \equiv 2$ is not choosable. On the other hand, one can show $f \equiv 3$ is choosable using a greedy algorithm. In other words, it is impossible to come up with lists of size 3 from which there is no proper coloring.

If (G, f) is choosable with $f \equiv k$ for some integer k, then G is said to be *k-choosable*. The smallest constant k for which is G is k-choosable, is called the *list chromatic number* or *choice number*, and has been a topic of considerable interest. It is typically denoted by $\chi_l(G)$. If all the lists are the same, list coloring reduces to ordinary coloring, so $\chi_l(G) \geq \chi(G)$. However, equality need not hold. The example in Figure 1.3 shows this, as $\chi(P_2 \,\square\, P_3) = 2$, but $\chi_l(P_2 \,\square\, P_3) = 3$. As another example, consider $K_{3,3}$. It is bipartite, hence $\chi(K_{3,3}) = 2$, but, in fact, $\chi_l(K_{3,3}) = 3$. To see this, let X and Y be the partite sets, and assign the lists 12, 13, and 23 to the three vertices of X, respectively, and also to the three vertices of Y (see Figure 1.4). No proper coloring from these lists can assign color 1 to one of the vertices of X and color 2 to another, as then there would be no way to properly color the vertex of Y assigned list 12. Similarly, no proper coloring could assign colors 1 and 3 or colors 2 and 3 to different vertices of X. Thus, there can be no proper coloring from these lists. Notice that the lists we chose are the 2-element subsets of a 3-element set. A similar argument using the k-element subsets of a $(2k-1)$-element set shows that $\chi_l(K_{p,p}) > k$ for $p = \binom{2k-1}{k}$. However, $\chi_l(K_{p,p}) \neq k+1$ if $k \geq 3$. For example, one can verify that the lists 123, 145, 167, 247, 256, 346, and 357 assigned to each partite set show that $K_{7,7}$ is not 3-choosable. These lists correspond to the lines of the Fano plane. In fact, for an arbitrary positive integer p, the choice number of $K_{p,p}$ is not known. Erdős, Rubin, and Taylor [3] showed that asymptotically $\chi_l(K_{p,p}) = \log_2 p + o(\log n)$.

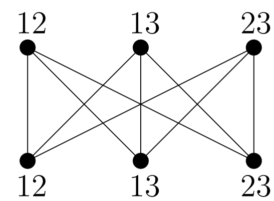


Figure 1.4: Lists showing $K_{3,3}$ is not 2-choosable

Even the determination of those pairs (p, q) for which $K_{p,q}$ is 3-choosable has turned out to be very difficult, though it has been solved. In fact, determination of the choice number on complete bipartite graphs is NP-Complete [8]. Further, A.L. Rubin showed that if f is a size function such that $f(V) \subset \{2, 3\}$, the problem of deciding if (G, f) is choosable is Π_2^p-Complete [3]. That is, if existing conjectures related to P=NP hold, then choosability is a strictly harder problem than ordinary coloring. Intuitively, graph coloring is hard in the sense that no polynomial algorithm is known (or is likely to be found) that will determine if an arbitrary graph is 3-colorable; however, given a 3-coloring of a graph, one can easily verify (in polynomial time) that it is indeed a 3-coloring. On the other hand, choosability is hard in the sense that no polynomial algorithm is known (or is likely to be found) that will determine if an arbitrary graph is f-choosable, for $f(V) \subset \{2, 3\}$; however, even given an f-assignment which is not choosable, it is an NP-complete problem to determine that there is no proper coloring from the lists (see [9]).

List coloring was introduced by Erdős, Rubin, and Taylor [3] in 1979, and independently by Vizing [9] in 1976. The authors of [3] proved that a graph is 2-choosable if and only if its *core* is 2-choosable, where the core of a graph is the graph obtained by repeatedly deleting vertices of degree 1 until no such vertices remain. This is reasonable, as a vertex with degree 1 and list size 2 can always be properly colored. Further, they proved that the only cores that are 2-choosable are a single vertex, even cycles, and theta graphs of the form $\theta_{1,1,2k+1}$ for $k \geq 0$. In addition, they proved an analogue of Brook's Theorem, namely that $\chi_l(G) \leq \Delta(G)$ for any connected graph G that is neither complete nor an odd cycle. Since then, considerable work has been directed to the study of list coloring. One notable result is Thomassen's elegant proof that planar graphs are 5-choosable [7], which since $\chi(G) \leq \chi_l(G)$, provides a short proof of the 5-color theorem for planar graphs. However, there are non 4-choosable planar graphs [10]. Perhaps the most well-known conjecture is the so-called list coloring conjecture, which states that for any graph G, $\chi_l(L(G)) = \chi(L(G))$. That is, if we color edges instead of vertices, the list chromatic number and ordinary chromatic number are the same. See [1], [8], and [12] for more details. Incidently, these references are quite interesting survey articles on list coloring.

1.4 Sum list coloring

Rather than fixing all the lists to have the same size, as is done in the study of the list chromatic number, we could allow the list sizes to vary, and try to minimize the sum of the list sizes. That is, we seek the smallest constant k for which there exists a choosable f with $\sum_{v \in V} f(v) = k$. This constant is called the *sum choice number* of the graph, and is denoted by $\chi_{\mathrm{SC}}(G)$. We denote $\sum_{v \in V} f(v)$ by size(f), and a choosable f for which size(f) is as small as possible is called a *minimum choice function*. Sum list coloring was introduced by Isaak [5, 6], and further studied by Berliner, Bostelmann, Brualdi, and Deatt [2].

Essentially, we are looking for a smallest choice function. Our measure of smallness is the size of f, the sum of the list sizes. Showing $\chi_{\mathrm{SC}}(G) = k$ proceeds in two parts. We must exhibit a size function f of size k such that every f-assignment has a proper coloring, and for every g of size $k-1$, we must show there exists a g-assignment with no proper coloring. Note that it suffices to consider size functions of size $k-1$, for if there exists a choosable g' of size $k-t$ for $t > 1$, then we get a choosable g of size $k-1$ by picking an arbitrary vertex v_0 and letting $g(v_0) = g'(v_0) + t - 1$ and $g(v) = g'(v)$ for $v \neq v_0$.

As an example, we now show that $\chi_{\mathrm{SC}}(C_3) = 6$. Let the vertices be v_1, v_2, v_3. The size function f given by $f(v_i) = i$ for $i = 1, 2, 3$ is choosable by greedy coloring (see Figure 1.5a). Conversely, up to symmetry, the only size functions of size 5 are those shown in Figure 1.5b. The size function with adjacent vertices of list size 1 is clearly not choosable. The other size function is not choosable, as the list assignment which assigns the list 12 to the two vertices of list size 2 and the list 1 to the other vertex has no proper coloring.

Notice that C_3 is not 2-choosable, as it is not 2-colorable. On the other hand, C_3 is clearly 3-choosable. For a given graph G, the determination of both $\chi_l(G)$ and $\chi_{\mathrm{SC}}(G)$ are minimization problems. We see that in this example, in some sense, sum list coloring gives a stronger result than ordinary choosability. The smallest choice function given by ordinary choosability is $(f(v_1), f(v_2), f(v_3)) = (3, 3, 3)$, whereas sum list coloring gives $(1, 2, 3)$.

We can get an upper bound for the sum choice number as follows. Choose any ordering $v_1, \ldots, v_n$ of the vertices of G. Define a size function f by $f(v_i) = 1 + |\{v_j : i < j, \text{ and } v_iv_j \in E(G)\}|$ for $i = 1, \ldots, n$. Then size$(f) = n + e$, where n is the

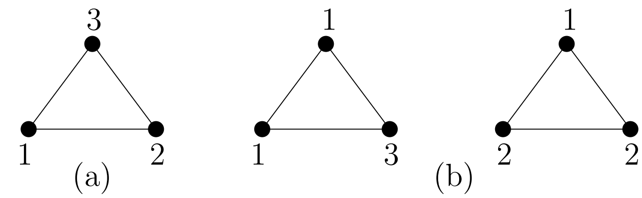


Figure 1.5: A choice function of size 6 on C_3, and the two nonchoosable size functions of size 5

number of vertices of G and e is the number of edges. Greedy coloring shows that f is choosable, and hence for any graph, $\chi_{\mathrm{SC}}(G) \leq n + e$. We will refer to $n + e$ as the *greedy bound*, and sometimes denote it by $\mathrm{GB}(G)$, or just GB when there is only one graph involved. Any graph for which the sum choice number is in fact $n + e$ is called *sc-greedy*. Isaak [6] showed that paths, cycles, complete graphs, and trees are all sc-greedy. In fact, he showed that *block graphs*, graphs in which every block is a cycle or a complete graph are sc-greedy. This result was improved by the authors of [2], who showed that if all the blocks of a graph are sc-greedy, then the graph itself is sc-greedy.

In addition to the aforementioned results, Isaak [5] determined the sum choice number of the Cartesian product $K_2 \square K_n$. The authors of [2] determined the sum choice number of the complete bipartite graph $K_{2,q}$, showing that the sum choice number is far below the greedy bound for large q. Moreover, they considered graphs obtained from K_n by joining a new vertex to all vertices of some subgraph, and showed that such graphs are sc-greedy. From this they concluded that every tree on n vertices can be obtained from the complete graph K_n by repeatedly deleting edges such that, at every step, the intermediate graphs are sc-greedy.

The most general and most difficult list coloring problem is to give a choosability characterization. That is, given a graph G, one seeks necessary and sufficient conditions on size functions f such that (G, f) is choosable, or perhaps a fast algorithm to decide if (G, f) is choosable. As mentioned before, deciding the choosability of a size function when all list sizes are 2 or 3 is Π_2^p-Complete. However, the problem is tractable for certain classes of graphs. For example, a size function on the path P_n is choosable if and only if there is no subgraph of P_n to which f assigns list size 1 to the two endvertices and list size 2 to every internal vertex (see Lemma 4.2). In addition, Isaak [6] provided some choosability tests for block graphs.

1.5 Thesis outline

In Chapter 2, we first prove a lemma that breaks up the computation of the sum choice number into two parts, allowing for a considerable simplification of many proofs. We then answer a few questions raised in [2]. The first of these is to show that $P_2 \square P_n$

is sc-greedy. Then, thinking of $P_2 \,\square\, P_n$ as a string of squares laid end-to-end, we consider, by analogy, a string of triangles laid end-to-end, and show it is also sc-greedy, though by a more delicate method. In answer to a second question, we show that the Peterson graph is sc-greedy. As a lemma required for the proof, we determine the sum choice number of theta graphs. We then partially answer another question by computing the sum choice number of $K_{3,q}$. The techniques developed are applicable to other graphs, and in fact we use them to provide a shorter computation of the sum choice number of $K_{2,n}$ than that given in [2]. In Chapter 3, we develop the idea of the *configuration number*, and prove a few lemmas of general applicability. We then use the ideas developed therein to compute the sum choice number of $P_3 \,\square\, P_n$. The proof is a lengthy case analysis. In Chapter 4, we consider choosability of fan graphs. In particular, Isaak, and independently Albertson and Pelsmajer, asked if every outerplanar graph is sc-greedy. We show that fan graphs, a class of outerplanar graphs, are not sc-greedy in general, and that, in fact, the gap between the greedy bound and the sum choice number can be arbitrarily large. The techniques used in Chapter 4 also have applications to other classes of graphs.

Chapter 2

Sum Choice Numbers of a Few Graphs

In this chapter, we present a few preliminary lemmas that will be used quite often. We then answer three questions raised in [2]. Namely, we determine the sum choice number of the Cartesian product $P_2 \square P_n$, the Peterson graph, and the complete bipartite graph $K_{3,q}$.

2.1 Preliminary results

Let G be a graph, and for a given induced subgraph H, we denote by f_H, $\mathcal{C}_H$, and c_H, the restrictions of the size function, etc. to H. Given a pair (G, f), we say that a vertex v is *forced* by an f-assignment $\mathcal{C}$ if it receives the same color in any proper $\mathcal{C}$-coloring of G. For any vertex $v \in V(G)$, we define the size function f^v on $G - v$ by $f^v(w) = f(w) - 1$, if w is adjacent to v, and $f^v(w) = f(w)$ otherwise. In addition, we define

$$\rho(G) = \min\{\chi_{\mathrm{SC}}(G - v) + \deg(v) + 1 : v \in V(G)\},$$
$$\tau(G) = \min\{\mathrm{size}(f) : f \text{ is choosable, and } 2 \leq f(v) \leq \deg(v)\, \forall v \in V(G)\}.$$

We call size functions f for which $f(v) = 1$ or $f(v) > \deg(v)$ for some vertex v, *simple size functions*, and all others, *non-simple size functions*. The following lemma is the simplest special case of Lemmas 7 and 8 in [6].

Lemma 2.1. *Let (G, f) be given.*

(a) If $f(v) = 1$ for some vertex $v \in V(G)$, then (G, f) is choosable if and only if $(G - v, f^v)$ is choosable.

(b) If $f(v) > \deg(v)$ for some vertex v, then (G, f) is choosable if and only if $(G - v, f_{G-v})$ is choosable.

Lemma 2.2. *For any graph G, $\chi_{\mathrm{SC}}(G) = \min\{\rho(G), \tau(G)\}$. In particular, if $G - v$ is sc-greedy for every $v \in V(G)$, then $\chi_{\mathrm{SC}}(G) = \min\{GB(G), \tau(G)\}$.*

Proof. Define

$$\begin{aligned}\alpha_1 &= \min\{\text{size}(f) : f \text{ is choosable and } f(v) = 1 \text{ for some } v \in V(G)\},\\ \alpha_2 &= \min\{\text{size}(f) : f \text{ is choosable and } f(v) > \deg(v) \text{ for some } v \in V(G)\}.\end{aligned}$$

Clearly, $\chi_{\mathrm{SC}}(G) = \min\{\tau(G), \alpha_1, \alpha_2\}$. Let f be a choice function on G. Note first that if $f(v) = 1$ for some $v \in V(G)$, then f^v is choosable by Lemma 2.1, and moreover, $\text{size}(f) = \text{size}(f^v) + \deg(v) + 1$. If $f(v) > \deg(v)$ for some $v \in V(G)$, then f_{G-v} is choosable by Lemma 2.1, and moreover, $\text{size}(f) = \text{size}(f_{G-v}) + f(v) \geq \text{size}(f_{G-v}) + \deg(v) + 1$. Note further that both $\min\{\text{size}(f^v) : v \in V(G) \text{ and } f^v \text{ is choosable}\}$, and $\min\{\text{size}(f_{G-v}) : v \in V(G) \text{ and } f_{G-v} \text{ is choosable}\}$ are equal to $\chi_{\mathrm{SC}}(G - v)$. It follows that $\alpha_1 = \alpha_2 = \rho(G)$. $\square$

The preceding lemma allows for considerable simplification of many proofs. Simple size functions can be thought of as somewhat trivial and bothersome cases that need to be considered, and the lemma above is our attempt to dispense with much of the trouble.

Lemma 2.3. *Let G be a graph decomposable into blocks $G_1, \ldots, G_k$. Then*

$$\chi_{\mathrm{SC}}(G) = \sum_{j=1}^{k} \chi_{\mathrm{SC}}(G_j) - k + 1.$$

In particular, a graph all of whose blocks are sc-greedy, is itself sc-greedy.

The lemma above follows immediately from Theorem 2.3 in [2]. An easy corollary is that a graph obtained from an sc-greedy graph by appending a leaf is also sc-greedy. In fact, if G' is obtained from G by appending a leaf, then $\chi_{\mathrm{SC}}(G') = \chi_{\mathrm{SC}}(G) + 2$. For convenience, the following lemma summarizes some of the results of [2] and [6].

Lemma 2.4 ([2], [6]). *Paths, cycles, complete graphs, and trees are sc-greedy.*

2.2 Strings of cycles

The following theorem answers a question raised in [2]. Recall that the symbol $\square$ denotes the Cartesian product, and P_n is the path on n vertices.

Theorem 2.5. *The graph $P_2 \square P_n$ is sc-greedy; that is, $\chi_{\mathrm{SC}}(P_2 \square P_n) = 5n - 2$.*

Proof. Label the vertices as in Figure 2.1. For any $k = 1, \ldots, n$, let G_k be the subgraph induced by vertices t_k and b_k, let L_k be the subgraph induced by vertices $t_1, b_1, \ldots, t_k, b_k$, and let R_k be the subgraph induced by vertices $t_k, b_k, \ldots, t_n, b_n$.

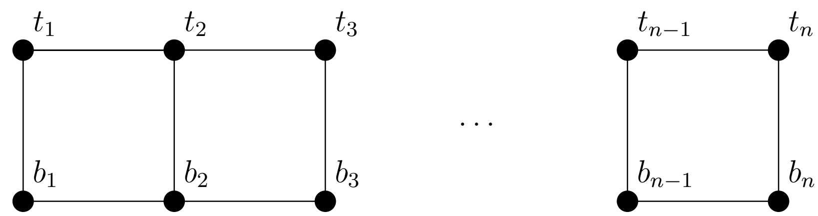


Figure 2.1: $P_2 \,\square\, P_n$

The proof is by induction on n. The basis P_2 is sc-greedy by Lemma 2.4. Now assume that $P_2 \,\square\, P_k$ is sc-greedy for $k < n$. We will show $G = P_2 \,\square\, P_n$ is sc-greedy; that is, its sum choice number is $5n-2$. Let f be a size function on G of size $5n-3$. We must show that f is not choosable. It is easy to see that if $\text{size}(f_{G_k}) \leq 2$ for any $k = 1, \dots, n$, then f is not choosable, so we may assume that $\text{size}(f_{G_k}) \geq 3$.

Assume that $\text{size}(f_{G_k}) \leq 4$ for some $1 < k < n$. By the induction hypothesis, we have $\chi_{\text{SC}}(L_k) = 5k-2$, and hence, $\text{size}(f_{L_{k-1}}) \geq 5k-6$. Similarly, by the induction hypothesis, $\chi_{\text{SC}}(R_k) = 5(n-k+1)-2$, so $\text{size}(f_{R_{k+1}}) \geq 5(n-k)-1$. Thus, if f is choosable, we must have $\text{size}(f) \geq (5k-6)+4+(5(n-k)-1) = 5n-3$, and hence, the above inequalities must be equalities. We will now define an uncolorable f-assignment $\mathcal{C}$. It is easy to check that since $\text{size}(f_{G_k}) \leq 4$, there exists an f_{G_k}-assignment $\mathcal{C}'$ such that there are at most two distinct proper $\mathcal{C}'$-colorings, c_1 and c_2, of G_k. Let $c_1 = c_2$ if there is only one. Let g_1 be a size function on L_{k-1} defined by $g_1(v) = f(v) - 1$ if $v \in V(G_{k-1})$ and $g_1(v) = f(v)$ otherwise, and let g_2 be a size function on R_{k+1} defined by $g_2(v) = f(v) - 1$ if $v \in V(G_{k+1})$ and $g_2(v) = f(v)$ otherwise. As $\text{size}(g_1) < \chi_{\text{SC}}(L_{k-1})$ and $\text{size}(g_2) < \chi_{\text{SC}}(R_{k+1})$, neither g_1 nor g_2 are choosable, hence there exists a g_1-assignment $\mathcal{C}_1$, and a g_2-assignment $\mathcal{C}_2$, neither of which have a proper coloring. Moreover, we may name the colors so that $\mathcal{C}'(G_k)$ is disjoint from $\mathcal{C}_1(L_{k-1})$ and $\mathcal{C}_2(R_{k+1})$. Define $\mathcal{C}$ by $\mathcal{C} = \mathcal{C}'$ on G_k, $\mathcal{C} = \mathcal{C}_1$ on L_{k-1}, and $\mathcal{C} = \mathcal{C}_2$ on R_{k+1}, except that we append $c_1(t_k)$, $c_1(b_k)$, $c_2(t_k)$, and $c_2(b_k)$ to $\mathcal{C}_1(t_{k-1})$, $\mathcal{C}_1(b_{k-1})$, $\mathcal{C}_2(t_{k+1})$, and $\mathcal{C}_2(b_{k+1})$, respectively. Let c be a $\mathcal{C}$-coloring. If c is proper, then c_{G_k} is equal to either c_1 or c_2. If $c_{G_k} = c_1$, then $c_{L_{k-1}}$ must be a proper $\mathcal{C}_1$-coloring of L_{k-1}, and if $c_{G_k} = c_2$, then $c_{R_{k-1}}$ must be a proper $\mathcal{C}_2$-coloring of R_{k-1}, neither of which exist. Hence $\text{size}(f_{G_k}) \geq 5$ for $k = 2, \dots, n-1$.

If $f(v_1) = 1$, then by Lemma 2.1, (G, f) is choosable if and only if $(G - v_1, f^{v_1})$ is. However, $G - v_1$ is sc-greedy by the induction hypothesis and the comment following Lemma 2.3, hence $\chi_{\text{SC}}(G - v_1) = 5n - 5 > 5n - 6 = \text{size}(f^{v_1})$, so f is not choosable. A similar argument applies if any of v_2, v_{n-1}, or v_n has list size 1. Thus $\text{size}(f_{G_1}) \geq 4$ and $\text{size}(f_{G_n}) \geq 4$, and hence, $\text{size}(f) = \sum_{k=1}^{n} \text{size}(f_{G_k}) \geq 5(n-2)+2(4) = 5n-2$. $\square$

In addition to the above result, we have determined by a lengthy case analysis that $P_3 \,\square\, P_n$ has sum choice number $\text{GB} - \lfloor n/3 \rfloor$ (see Chapter 3 for details). Moreover, ideas very similar to those used in proof above show that if instead of 4-cycles, we were to use cycles of arbitrary and varying lengths greater than 3, the graph obtained would still be sc-greedy. In fact, if instead of merely laying cycles end-to-end, we were to lay them along a tree structure or along a cycle, the resulting graph would still be sc-greedy. However, 3-cycles are somewhat more complicated to deal with. Consider the graph Tr(t), pictured in Figure 2.2, obtained by laying t triangles end-to-end. Formally, Tr(t) has vertex set $\{v_1, \ldots, v_{t+2}\}$ with v_i adjacent to v_j if and only if $0 < |i - j| \leq 2$. Below we prove that Tr(t) is sc-greedy. A longer proof using the same techniques can be used to show that for any minimum choice function f on Tr(t), there exists an f-assignment forcing the vertices v_1 and v_2.

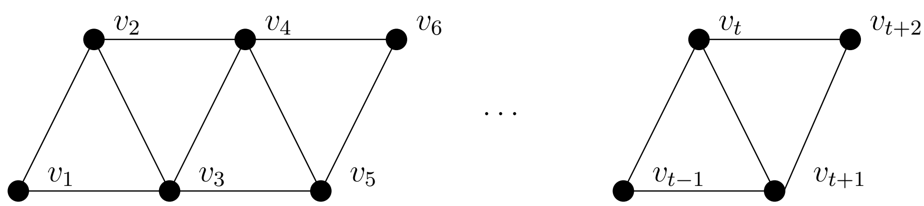


Figure 2.2: Tr(t)

Theorem 2.6. *The graph Tr(t) is sc-greedy; that is, $\chi_{\mathrm{SC}}(Tr(t)) = 3t + 3$.*

Proof. Let R_k denote the subgraph of Tr(t) which is induced by the vertices $v_k, \ldots, v_{t+2}$. We will prove by induction on t that Tr(t) is sc-greedy; that is, it has sum choice number $3t + 3$. The basis Tr(1) is a complete graph, and is therefore sc-greedy by Lemma 2.4. Now assume that Tr(s) is sc-greedy for all $s < t$. Note that removing a vertex from Tr(t) leaves a graph whose blocks are either paths or copies of Tr(s) for values of s less than t. By the induction hypothesis and Lemma 2.3, such a graph is sc-greedy. Thus by Lemma 2.2, it remains to show $\tau(\mathrm{Tr}(t)) \geq 3t + 3$.

We must now show that any non-simple size function f of size $3t + 2$ is not choosable. Note that $f(v_1) = 2$ since $\deg(v_1) = 2$. Let i_0 be the least index greater than 1 of a vertex having list size 2. This index must exist and be at most t, as otherwise size(f) would exceed $3t + 2$. As $i_0 \leq t$, R_{i_0} and R_{i_0+1} are defined. Note that if there is a vertex v_k, $k \leq i_0$, with $f(v_k) \geq 4$, then $\mathrm{size}(f_{R_{i_0}}) < \chi_{\mathrm{SC}}(R_{i_0})$. Thus we may assume that $f(v_1) = f(v_{i_0}) = 2$ and $f(v_k) = 3$ for $1 < k < i_0$. We now create an f-assignment $\mathcal{C}$ with no proper coloring. Let $\mathcal{C}(v_0) = 12$, and $\mathcal{C}(v_i) = 123$ for $1 < i < i_0$. Let $\mathcal{C}(v_{i_0})$ be 34 if i_0 is congruent to 1 modulo 3, and $\mathcal{C}(v_{i_0}) = 12$ otherwise. It can be checked that no proper $\mathcal{C}$-coloring can use color 3 on v_i for each $i \leq i_0$ congruent to 1 modulo 3. We will define $\mathcal{C}$ on R_{i_0+1} differently according to the congruence of i_0 modulo 3.

If $i_0 \equiv 0 \pmod 3$, let g be size function on R_{i_0} given by $g(v_{i_0+1}) = f(v_{i_0+1}) - 1$, and let g agree with f elsewhere. Since $\mathrm{size}(g) < \chi_{\mathrm{SC}}(R_{i_0})$, g is not choosable. Let $\mathcal{C}'$ be an uncolorable g-assignment with $\mathcal{C}'(v_{i_0}) = 12$ and $3 \notin \mathcal{C}'(v_{i_0+1})$. Define $\mathcal{C}$ on

R_{i_0+1} by letting $\mathcal{C}$ equal $\mathcal{C}'$, except that we append color 3 to the list for v_{i_0+1}. Any proper coloring cannot use color 3 on v_{i_0-2}, and hence must use color 3 on v_{i_0+1}, since both colors 1 and 2 must be used on neighbors of v_{i_0+1}. Thus we must color R_{i_0} from C', which is not possible.

If $i_0 \equiv 1 \pmod 3$, let g be the size function on R_{i_0+1} given by $g(v_{i_0+1}) = f(v_{i_0+1})-1$, $g(v_{i_0+2}) = f(v_{i_0+2})-1$, and let g agree with f elsewhere. Since $\text{size}(g) < \chi_{\text{SC}}(R_{i_0+1})$, g is not choosable. Let $\mathcal{C}'$ be an uncolorable g-assignment such that color 4 does not appear in $C'(v_{i_0+1})$ nor in $C'(v_{i_0+2})$. Define $\mathcal{C}$ on R_{i_0+1} by letting $\mathcal{C}$ equal $\mathcal{C}'$, except that we append color 4 to the lists for v_{i_0+1} and v_{i_0+2}. Any proper coloring must not use color 3 on v_{i_0}, and hence must use color 4 there. So any proper coloring must not use color 4 on v_{i_0+1} and v_{i_0+2}, and hence, we must color R_{i_0+1} from $\mathcal{C}'$, which is not possible.

Finally, if $i_0 \equiv 2 \pmod 3$, let g be the size function on R_{i_0+1} given by $g(v_{i_0+1}) = f(v_{i_0+1}) - 2$ and let g agree with f elsewhere. Since $\text{size}(g) < \chi_{\text{SC}}(R_{i_0+1})$, g is not choosable. Let $\mathcal{C}'$ be an uncolorable g-assignment such that neither color 1 nor color 2 appears on $\mathcal{C}'(v_{i_0+1})$. Define $\mathcal{C}$ on R_{i_0+1} by letting $\mathcal{C}$ equal $\mathcal{C}'$, except that we append colors 1 and 2 to the list for v_{i_0+1}. A proper $\mathcal{C}$-coloring must not use color 3 on v_{i_0-1}, and hence, neither color 1 nor color 2 can be used on v_{i_0+1}, since both colors must be used on its neighbors. Thus we must color R_{i_0+1} from $\mathcal{C}'$, which is not possible. □

2.3 Theta graphs and the Peterson graph

The authors of [2] asked about the choice number of the Peterson graph. To resolve this question, we will need the following lemma. Recall that a theta graph, θ_{k_1,k_2,k_3}, is a simple graph consisting of two vertices connected by three internally vertex disjoint paths, having k_1, k_2, and k_3 internal vertices, respectively, $0 \le k_1 \le k_2 \le k_3$. Recall that we denote the greedy bound, the sum of the number of vertices and edges, by GB, which in this case is $2(k_1 + k_2 + k_3) + 5$.

Theorem 2.7. $\chi_{\text{SC}}(\theta_{k_1,k_2,k_3}) = \begin{cases} \text{GB}-1, & \textit{if } k_1 = k_2 = 1 \textit{ and } k_3 \textit{ is odd} \\ \text{GB}, & \textit{otherwise.} \end{cases}$

Proof. Removing a vertex from a theta graph leaves either a tree or a cycle with pendant edges, both of which are sc-greedy. Hence, by Lemma 2.2, it remains to determine $\tau(\theta_{k_1,k_2,k_3})$. If f is a non-simple size function with $\text{size}(f) = \text{GB}-1 = 2(k_1 + k_2 + k_3) + 4$, then $f \equiv 2$, since the vertex set has size $k_1 + k_2 + k_3 + 2$. However, by a well-known result in [3], the only theta graphs that are 2-choosable have $k_1 = k_2 = 1$ and k_3 odd. □

By Lemma 2.2, to show a graph G is sc-greedy, it suffices to show that for any vertex v of G, $G - v$ is sc-greedy, and then to show that there is no choosable non-simple size function of size one less than the greedy bound. The same, of course, applies to $G - v$, so we get a recursive procedure whereby we remove vertices from G

until we get to graphs we know are sc-greedy, and at each stage we show that there are no choosable non-simple size functions of size one less than the greedy bound.

The following will be important in the proof below: Odd cycles are not 2-choosable, because the list assignment with all lists equal to 12 has no proper coloring. Moreover, this implies that if we assign lists 12 to all vertices of an odd cycle but one, which gets list 123, then color 3 must be used on that vertex.

Theorem 2.8. *The Peterson Graph is sc-greedy; that is, it has sum choice number 25.*

Proof. Denote the Peterson graph by P, let Q denote P minus a vertex, and let R denote Q minus a vertex of degree 2. The greedy bound is 25 for P, 21 for Q, and 18 for R (see Figure 2.3).

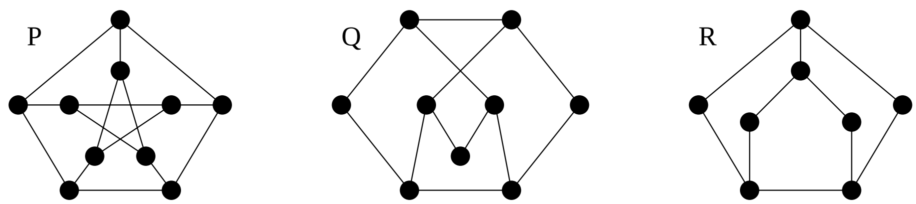


Figure 2.3: The graphs P, Q, and R of Theorem 2.8

Recall that attaching a leaf to an sc-greedy graph produces an sc-greedy graph by the comment following Lemma 2.3. For any vertex $v \in V(R)$, $R - v$ is either an sc-greedy theta graph or a cycle with pendant edges, and hence sc-greedy, and for any vertex $v \in V(Q)$ of degree 3, $Q - v$ is an sc-greedy theta graph with pendant edges. It remains to consider non-simple size functions of size one less than the greedy bound on each of P, Q, and R.

The only non-simple size functions of size 17 on R assign list size 2 for all but one vertex, hence there is a 5-cycle all of whose list sizes are 2, which is not colorable. The only non-simple size functions of size 20 on Q assign list size 2 to all but two vertices v and w, both having degree 3. It can be checked that there is a 5-cycle avoiding any pair of adjacent vertices, and hence, if v and w are adjacent, then there is a 5-cycle all of whose list sizes are 2, which is not colorable. If, on the other hand, v and w are not adjacent, then they must be at distance two from each other. Let x denote their common neighbor. It can be checked that there exist 5-cycles C_1 and C_2 with v in C_1, but not C_2, w in C_2, but not C_1, and x not in either. Let f be a non-simple size function of size 20, and create an f-assignment $\mathcal{C}$ with $\mathcal{C}(v) = 123$, $\mathcal{C}(w) = 124$, $\mathcal{C}(x) = 34$, and let any other vertex have list 12. These lists force color 3 on v and color 4 on w. Hence, there is no proper $\mathcal{C}$-coloring because $\mathcal{C}(x) = 34$.

Thus, it remains to consider non-simple size functions of size 24 on P. Any such size function assigns list size 3 to four vertices, and list size 2 to all others. Let H denote the subgraph induced by the vertices assigned list size 3. We consider cases according to the possibilities for H.

If H is a path, P_4, then it can be checked that there exists a 5-cycle all of whose list sizes are 2, which is not colorable. The other possibilities are that H is: (a) a claw, $K_{1,3}$, (b) a 3-path union a vertex, $P_3 + K_1$ (c) two disjoint paths, $K_2 + K_2$, (d) a path and two isolated vertices $K_2 + K_1 + K_1$, or (e) 4 isolated vertices, $K_1 + K_1 + K_1 + K_1$. See Figure 2.4 for lists showing in each case that the size function is not choosable. The vertices of H are indicated with solid circles, and vertices with no list specified can have any list. Note that by symmetry, these pictures give the only layouts of H that need be considered.

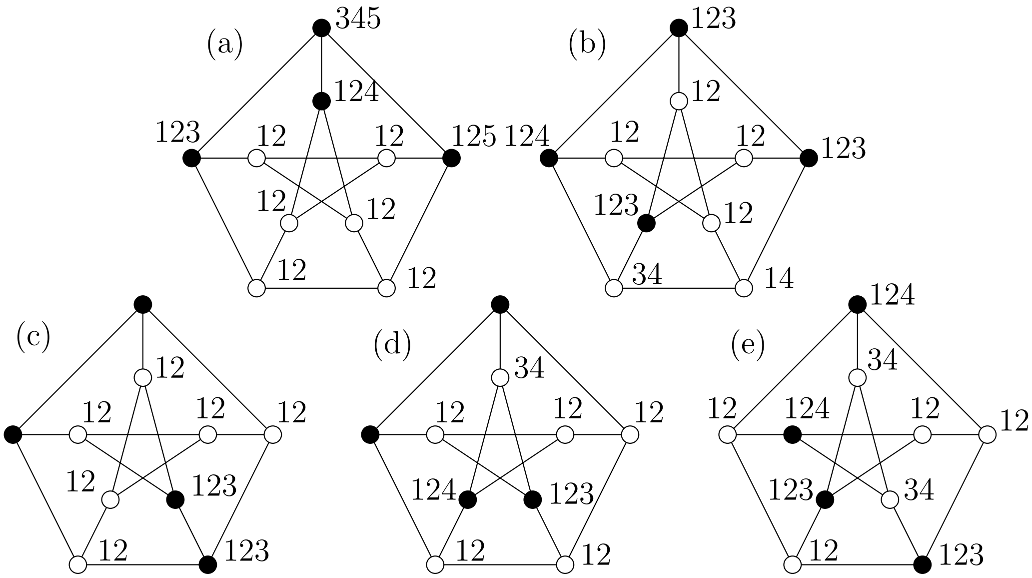


Figure 2.4: The lists for the cases (a) through (e) of Theorem 2.8

For (a), the vertices with lists 123, 124, and 125 are each contained in different 5-cycles whose other vertices have lists 12, thus colors 3, 4, and 5, respectively, are forced on those vertices, and hence, the top vertex can't be colored. For (b), the inside star-shaped cycle forces color 3 on the isolated vertex of H. From there it is easy to check that these lists are not colorable.

For (c), color 3 is forced on both of the vertices with list 123 as both are contained in 5-cycles whose other vertices all have list 12. As these vertices are adjacent, the lists are not colorable. For (d), colors 3 and 4 are forced on the vertices with lists 123 and 124, respectively, as both vertices are contained in 5-cycles whose other vertices have lists 12. The vertex with list 34 is adjacent to both of these vertices, and hence can't be colored.

For (e), we will have to consider two cases. First suppose a proper coloring uses color 3 on the bottom right vertex. Then color 4 must be used on the inside vertex adjacent to it with list 34, which implies that color 3 must be used on the other inside vertex with list 34. Hence the inside vertices with list 123 and 124 cannot be colored with colors 3 and 4, respectively. However any proper coloring must use color 3 or color 4 on one of these two vertices, as otherwise the 5-cycle containing those two vertices and three vertices with list 12 would not be colorable. Thus no proper coloring chooses color 3 on the bottom right vertex, so we may regard it henceforth as having list 12. From here we conclude that colors 3 and 4 are forced on the leftmost vertex with list 123 and the topmost vertex with list 124, respectively, as both vertices are contained in 5-cycles whose other vertices all have list 12. These two vertices have a common neighbor with list 34, which then cannot be colored. $\square$

2.4 Complete bipartite graphs

Let (G, f) be given, and let $\mathcal{C}$ be an f-assignment. Let Y be an independent set in G, and let X be the subgraph of G induced by the vertices not in Y. Let c be a proper $\mathcal{C}_X$-coloring. For any $y \in V(Y)$, form a set $A(c, y)$ from the distinct colors used by c on $N(y)$, the neighborhood of y. We will say c is *blocked* by Y if there is a vertex $y \in V(Y)$ such that $\mathcal{C}(y) \subset A(c, y)$. In other words, c cannot be extended to a proper $\mathcal{C}$-coloring of all of G. We then have the following.

Lemma 2.9. *Let (G, f) be given, let $\mathcal{C}$ be an f-assignment, Let Y be an independent set in G, and let X be the subgraph induced by the vertices of G not in Y. Then $\mathcal{C}$ is colorable if and only if there exists a proper $\mathcal{C}_X$-coloring that is not blocked by Y.*

Proof. Let $\widetilde{c}$ be a proper $\mathcal{C}_X$-coloring not blocked by Y. We need to define a proper $\mathcal{C}$-coloring c. For each y in $V(Y)$, let $c(y)$ be any color in $\mathcal{C}(y) \backslash A(\widetilde{c}, y)$, which exists since $\widetilde{c}$ is not blocked by Y. Conversely, if every proper $\mathcal{C}_X$-coloring is blocked by Y, then there is no proper $\mathcal{C}$-coloring, since for any $\mathcal{C}$-coloring c, there exists a vertex $y \in V(Y)$ for which there is no color in $\mathcal{C}(y)$ not used by c on a neighbor of y. $\square$

It then follows from the definition of choosability that f is choosable if and only if for every f-assignment $\mathcal{C}$, some proper $\mathcal{C}_X$-coloring is not blocked by Y. We can apply these ideas to the computation of the sum choice number of the complete bipartite graph $K_{p,q}$, $p \leq q$. Let X and Y denote the partite sets of size p and q, respectively, with $V(X) = \{x_1, \ldots, x_p\}$. Let $\alpha(p, q)$ be the minimum size of a choosable f with $f_Y \equiv 2$, let $\beta(p, q)$ be the minimum size of a choosable f with $f(y) \in \{2, \ldots, p\}$ for all $y \in V(Y)$, and let $\gamma(p, q)$ be the minimum size over all other choosable f. Clearly, $\chi_{\mathrm{SC}}(K_{p,q})$ is given by the minimum of these three values. By the ideas in the proof of Lemma 2.2, $\gamma(p, q) = \chi_{\mathrm{SC}}(K_{p,q-1}) + p + 1$. We will use the blocking idea to compute $\alpha(p, q)$ for a fixed p. Let f be a size function on $K_{p,q}$ with $f_Y \equiv 2$, and let $\mathcal{C}$ be an f-assignment. Since X is an independent set, the collection of all proper $\mathcal{C}_X$-colorings can be identified with all p-tuples $(a_1, \ldots, a_p)$, with $a_i \in \mathcal{C}(x_i)$ for each $i = 1, \ldots, p$. Since $N(y) = X$ for every $y \in V(Y)$, there exists a set A_c such that $A(c, y) = A_c$ for all $y \in V(Y)$. Thus a proper $\mathcal{C}_X$-coloring c is blocked if and only if $\mathcal{C}(y) \subset A_c$ for some $y \in V(Y)$. The sum choice number of $K_{2,q}$ was determined in [2]. We provide a somewhat similar proof here which will generalize to $K_{3,q}$.

Theorem 2.10 (Berliner et al.). *The sum choice number of $K_{2,q}$ is given by*

$$\chi_{\mathrm{SC}}(K_{2,q}) = 2q + \min\{l + m : q < lm, \text{ with } l, m \in \mathbb{N}\}.$$

Proof. We will compute $\alpha(2,q)$ and then show $\alpha(2,q) \leq \gamma(2,q)$. Note that $\alpha(2,q) = \beta(2,q)$. Fix positive integers l and m. Consider a size function f on $K_{2,q}$ with $f(x_1) = l$ and $f(x_2) = m$, and $f_Y \equiv 2$. Using the blocking idea, if $\mathcal{C}$ is an f-assignment such that there exists a color a in $\mathcal{C}(x_1) \cap \mathcal{C}(x_2)$, then we get a proper $\mathcal{C}$-coloring by coloring x_1 and x_2 with a, since there can be no 2-set contained in $\{a\}$. If the lists on X are disjoint, then there are a total of lm proper colorings from the lists on X, and each vertex of Y can be used to block exactly one of them. Thus if $q < lm$, there is always some proper coloring not blocked, whereas if $q \geq lm$, there exists a list assignment blocking every proper coloring. We conclude that $\alpha(2,q) = 2q + \min\{l + m : q < lm, \text{ with } l, m \in \mathbb{N}\}$.

We will now show $\alpha(2,q) \leq \gamma(p,q) = \chi_{\mathrm{SC}}(K_{2,q-1}) + 3$ by induction. For the base case, $\alpha(2,1) = 5 = \chi_{\mathrm{SC}}(K_{2,0}) + 3$. Now assume the inequality holds for $q-1$. Then $\chi_{\mathrm{SC}}(K_{2,q-1}) = \alpha(2,q-1)$. Hence, the inequality for q holds if and only if $M \leq N+1$, where $M = \min\{l + m : q < lm\}$ and $N = \min\{l + m : q - 1 < lm\}$, with both minimums taken over positive integers. Pick (l^*, m^*) giving the minimum, N. Then $q < l^*m^* + 1 \leq l^*(m^* + 1)$. Hence, $M \leq l^* + (m^* + 1) = N + 1$. □

The proof above and Lemma 2.1 combine to give a characterization of choosability for $K_{2,q}$. Let $u = |f^{-1}(1) \cap V(Y)|$ and $d = |f^{-1}(2) \cap V(Y)|$. It is straightforward to show that f is choosable if and only if $d < (f(x_1) - u)(f(x_2) - u)$.

Corollary 2.11. *Explicitly, the sum choice number of $K_{2,q}$ is given by*

$$\chi_{\mathrm{SC}}(K_{2,q}) = 2q + 1 + \lfloor \sqrt{4q+1} \rfloor.$$

Proof. We compute $\alpha(2,q)$ explicitly. Suppose $(l,m) = (l, l+t)$ for $t > 1$. Then if $q < lm$, we have $q < l(l+t) \leq l^2 + lt + t - 1 = (l+1)(l+t-1)$, hence the minimum, $\alpha(2,q)$, must occur at (l,m) of the form (l,l) or $(l,l+1)$. Define functions $g_1(k) = \lfloor (\sqrt{4k+1} - 1)/2 \rfloor$ and $g_2(k) = \lfloor \sqrt{k} \rfloor$. These are nondecreasing over $\mathbb{N}$ and satisfy $g_1(l(l+1)) = g_2(l^2) = l$ for all $l \in \mathbb{N}$. Suppose (l^*, m^*) gives the minimum, $\alpha(2,q)$. Then l^* must satisfy the inequality $(l^* - 1)l^* \leq q < l^*(l^* + 1)$, and applying g_1 to this gives $l^* = g_1(q) + 1$. Similarly, m^* must satisfy the inequality $(m^* - 1)^2 \leq q < (m^*)^2$, and applying g_2 to this gives $m^* = g_2(q) + 1$. Thus $\chi_{\mathrm{SC}}(K_{2,q}) = 2q + 2 + g_1(q) + g_2(q)$, and this quantity is equal to $2q + 1 + \lfloor \sqrt{4q+1} \rfloor$. To see this, let $r = \lfloor \sqrt{4q+1} \rfloor$ and $s = \lfloor \sqrt{4q} \rfloor$. If $r = s$, it is easy to check that the two quantities are equal by considering r odd and even separately. If $r = s + 1$, then $4q + 1$ is an odd perfect square, hence we need only check that the two quantities are equal for odd r, which is easily seen to be true. □

These same techniques can be used to find the sum choice number of $K_{3,q}$. Let f be a size function on $K_{3,q}$ satisfying $f(y) \in \{2, 3\}$ for all $y \in V(y)$, $f(x_1) = l$,

$f(x_2) = m$, and $f(x_3) = n$, with $0 < l \le m \le n$, and let $t = |f^{-1}(3) \cap V(Y)|$. We will denote this by $f = (l, m, n : t)_q$. When we use this notation, it will be implicit that $f(y) \in \{2, 3\}$ for all $y \in V(Y)$. We provide an example here to motivate the proof of Theorem 2.12. In the proof of Theorem 2.10, we considered any size function f satisfying $f(x_1) = l$, $f(x_2) = m$ and $f_Y \equiv 2$. For the sake of illustration, suppose that $l = 2$ and $m = 3$. The only f-assignment of interest has disjoint lists on x_1 and x_2, say $\mathcal{C}(x_1) = 12$ and $\mathcal{C}(x_1) = 345$. We could be certain that every proper coloring is blocked, provided we assign the lists 13, 14, 15, 23, 24, 25 on Y. If any of these lists is missing, then there exists a proper $\mathcal{C}$-coloring. Hence, we conclude that f is choosable if and only if $q < 6$.

For $K_{3,q}$ things are complicated by the fact that there are now list-assignments of interest where the lists are not all disjoint. Consider the size function $(4, 4, 4 : 0)_q$. If we put lists 1234, 5678, and *abcd* on the vertices of X, it turns out that 16 2-sets is the minimum number needed to block every proper coloring of X, namely all 2-sets with one element coming from $\{1, 2, 3, 4\}$ and the other from $\{5, 6, 7, 8\}$. If instead we put lists 1234, 1256, and 3456 on X, then only 12 2-sets are needed to block every proper coloring, namely 13, 14, 15, 16, 23, 24, 25, 26, 35, 36, 45, and 46. It turns out that these X-lists are worst-possible in the sense that they require the least number of 2-sets to block every proper coloring. That is, regardless of the collection of size 4 lists we put on X, if there are less than 12 vertices in Y, then there is always a proper coloring of the entire graph. Hence we conclude that $(4, 4, 4 : 0)_q$ is choosable if and only $q < 12$. By finding the worst possible lists for any $l \le m \le n$ we get a quantity, $q^*(l, m, n)$, which gives the minimum value of q such that $(l, m, n : 0)_q$ is not choosable. Thus we conclude that $\alpha(3, q) = 2q + \min\{l + m + n : q < q^*(l, m, n)\}$, with the minimum taken over $l, m, n \in \mathbb{N}$. Certain properties of $q^*(l, m, n)$ will allow us to show that $\alpha(3, q) \le \beta(3, q)$, and a similar argument to the one used in Theorem 2.10 will show $\alpha(3, q) \le \gamma(3, q)$.

Theorem 2.12. *The sum choice number of $K_{3,q}$ is given by*

$$2q + \min\{l + m + n : q < q^*(l, m, n), \text{ with } l, m, n \in \mathbb{N},\ l \le m \le n\},$$

where $q^*(l, m, n)$ is given by $lm - \lfloor (l + m - n)^2/4 \rfloor$ if $n \le l + m$, and by lm otherwise.

Proof. We will first compute $\alpha(3, q)$, then show $\beta(3, q) = \alpha(3, q)$, and finally show that $\alpha(3, q) \le \gamma(3, q)$. Fix positive integers l,m, and n, with $l \le m \le n$. Consider the size function $f = (l, m, n : 0)_q$, and let $\mathcal{C}$ be an f-assignment. We will determine the minimum number of 2-sets needed to block every proper $\mathcal{C}_X$-coloring. If there exists a color a in $\mathcal{C}(x_1) \cap \mathcal{C}(x_2) \cap \mathcal{C}(x_3)$, then we get a proper $\mathcal{C}$-coloring by coloring x_1, x_2, and x_3 with a, since there can be no 2-set contained in $\{a\}$. So assume there is no color in common to all the lists on X. In this case, $\mathcal{C}_X$ has the following form:

$$\begin{aligned} \mathcal{C}(x_1) &= a_1 \dots a_{k_1}\, b_1 \dots b_{k_2}\, c_1 \dots c_{k_4}, \\ \mathcal{C}(x_2) &= a_1 \dots a_{k_1}\, d_1 \dots d_{k_3}\, e_1 \dots e_{k_5}, \\ \mathcal{C}(x_3) &= b_1 \dots b_{k_2}\, d_1 \dots d_{k_3}\, f_1 \dots f_{k_6}. \end{aligned}$$

Colors with different names are distinct, and some of the k_i may be zero. In order to block each proper coloring, we require all 2-sets of the forms a_ib_j, a_id_j, a_if_j, b_id_j, b_ie_j, c_id_j, where i and j range over all possible values. The sets remaining unblocked are of the form $\{c_i, e_j, f_k\}$. To block these with the minimum number of 2-sets, add to the collection all 2-sets of the forms c_ie_j, c_if_j, or e_if_j, whichever gives the least number. In total, a smallest collection of 2-sets must have $k_1n + k_2(m-k_1) + k_3(l-k_1-k_2) + \min\{(l-k_1-k_2)(m-k_1-k_3), (l-k_1-k_2)(n-k_2-k_3), (m-k_1-k_3)(n-k_2-k_3)\}$, which simplifies to $\min\{\delta(l,m,n,k_1), \delta(l,n,m,k_2), \delta(m,n,l,k_3)\}$, where $\delta(x,y,z,w) = xy + w(z-x-y+w)$.

We now minimize this over all possible $\mathcal{C}_X$-assignments to find the list assignment requiring the least number of 2-sets to block every proper coloring. The minimum number of 2-sets needed in this case will be denoted by $q^*(l,m,n)$, which in fact gives the minimum value of q such that $(l,m,n : 0)_q$ is not choosable. To determine a formula for $q^*(l,m,n)$, we determine the minimum of the expression in the previous paragraph over all nonnegative integers k_1, k_2, and k_3 satisfying $l \geq k_1, k_2 \geq 0$ and $m \geq k_3 \geq 0$. Note that $\delta(l,m,n,k_1)$ is a quadratic function in k_1, and a simple analysis shows that the minimum occurs at $k_1 = \lfloor (l+m-n)/2 \rfloor$ for $n \leq l+m$ and $k_1 = 0$ for $n > l+m$. A similar analysis applies to the other two delta quantities, and it can be checked that the minimum obtained from each of the three delta quantities is equal to $lm - \lfloor (l+m-n)^2/4 \rfloor$ for $n \leq l+m$ and lm for $n > l+m$. This quantity is $q^*(l,m,n)$, and we conclude

$$\alpha(3,q) = \min\{l+m+n : q < q^*(l,m,n), \text{ with } l,m,n \in \mathbb{N},\ l \leq m \leq n\}.$$

Now we show $\beta(3,q) = \alpha(3,q)$. Note that this is clearly true when $q = 0$. For $q > 1$ we show that any size function $g = (l', m', n' : t)_q$ of size $\alpha(3,q) - 1$ with $t > 0$ is not choosable. We will assume on the contrary that g is choosable, and construct a sequence of size functions $h_i = (l_i, m_i, n_i : t-i)$ for $i = 0, \ldots t$, with $h_0 = g$, such that if h_i is choosable, then so is h_{i+1}, and then show that h_t is in fact not choosable, thereby contradicting our assumption that g is choosable. Let $d_i = |h_i^{-1}(2) \cap V(Y)|$. We may assume that $n' > 1$ as otherwise g must equal $(1,1,1 : t)_q$, which is not choosable for $q > 0$ and any t. Let $i_0 = n' - m'$ and let $h_0 = g$. For $i = 1, \ldots i_0$, let $l_i = l'$, $m_i = m_{i-1} + 1$, and $n_i = n'$. Let $i_1 = i_0 + 1$ if $l' = 1$, and i_0 if $l' > 1$. If $l' = 1$, let $l_{i_1} = 2$, $m_{i_1} = m_{i_0}$, and $n_{i_1} = n_{i_0}$. For $j \geq 1$, let $l_{i_1+j} = l_{i_1}$, $m_{i_1+j} = m_{i_1} + \lfloor j/2 \rfloor$ and $n_{i_1+j} = n_{i_1} + \lceil j/2 \rceil$. Now, for $l,m,n \in \mathbb{N}$ one can easily compute that $q^*(l,m,n)$ is strictly greater than both $q^*(l-1,m,n)$ and $q^*(l,m-1,n)$, and if $n < l+m$, then $q^*(l,m,n)$ is strictly greater than $q^*(l,m,n-1)$. Note that we have arranged it so that for each $i = 1, \ldots, t$, $l_i \leq m_i \leq n_i$, and for $i = i_1, \ldots, t$, $n_i < l_i + m_i$. Let $q_i = q^*(l_i, m_i, n_i)$. By assumption, h_0 is choosable. Now let $1 < i < t$ and assume h_i is choosable. Then $d_i < q_i$. Thus we have $d_{i+1} = d_i + 1 < q_i + 1 \leq q_{i+1}$, by the strict montonicity of q^* in each argument. Hence h_{i+1} is also choosable. Thus $h_t = (l_t, m_t, n_t : 0)$ is choosable and of size $\alpha(3,q) - 1$, a contradiction.

Finally, we show by induction that $\alpha(3,q) \leq \gamma(3,q) = \chi_{\mathrm{SC}}(K_{3,q-1}) + 4$. For the base case, $\alpha(3,1) = 7 = \chi_{\mathrm{SC}}(K_{3,0}) + 4$. Assume the inequality holds for $q-1$.

Then $\chi_{\mathrm{SC}}(K_{3,q-1}) = \alpha(3, q-1)$. Hence, the inequality for q holds if and only if $M \leq N+2$ where $M = \min\{l+m+n : q < q^*(l,m,n)\}$ and $N = \min\{l+m : q-1 < q^*(l,m,n)\}$, with both minimums taken over positive integers. Pick (l^*, m^*, n^*) giving the minimum, N. Then $q < l^*m^* - \lfloor \frac{(l^*+m^*-n^*)^2}{4} \rfloor + 1 \leq l^*(m^*+1) - \lfloor \frac{(l^*+(m^*+1)-(n^*+1))^2}{4} \rfloor$. Hence, $M \leq l^* + (m^*+1) + (n^*+1) = N$. □

Corollary 2.13. *Explicitly, the sum choice number of $K_{3,q}$ is given by*

$$\chi_{\mathrm{SC}}(K_{3,q}) = 2q + 1 + \lfloor \sqrt{12q+4} \rfloor.$$

Proof. We compute $\alpha(3, q)$ explicitly. Recall that for $n \geq l + m - 1$, $q^*(l,m,n) = lm = q^*(l, m, l+m-1)$, so we only need to consider $n < l + m$. Note that if l and the sum $m + n$ are each fixed constants, then $lm - \lfloor (l+m-n)^2/4 \rfloor$ is maximized when $n - m$ is as close to zero as possible, and on the other hand, if m and the sum $l + n$ are each fixed constants, then $lm - \lfloor (l+m-n)^2/4 \rfloor$ is maximized when $n - l$ is as close to zero as possible. Suppose that $(l,m,n) = (l, l+c, l+d)$, for $d \geq c \geq 0$, $d \geq 2$. Let (l', m', n') be equal to $(l, l+1, l+d-1)$ if $c = 0$, $(l+1, l+d-1, l+d)$ if $c = d$, and $(l+1, l+c, l+d-1)$ for any other value of c. Then (l', m', n') satisfies $l' + m' + n' = l + m + n$, and also $n' - m' < n - m$ if $c = 0$, and $n' - l' < n - l$ for any other value of c. Thus $q^*(l', m', n') > q^*(l, m, n)$. We conclude that to determine the above minimum, it suffices to consider only those (l, m, n) of the forms (l, l, l), $(l, l, l+1)$, and $(l, l+1, l+1)$.

Define functions $g_1(k) = \lfloor (\sqrt{12k+4} - 2)/3 \rfloor$, $g_2(k) = \lfloor (\sqrt{12k+4} - 1)/3 \rfloor$, and $g_3(k) = \lfloor \sqrt{12k}/3 \rfloor$. Note that these functions are nondecreasing over $\mathbb{N}$ and satisfy $g_1(q^*(l, l+1, l+1)) = g_2(q^*(l, l, l+1)) = g_3(q^*(l, l, l)) = l$ for all $l \in \mathbb{N}$. Let (l^*, m^*, n^*) give the minimum. To find l^*, note that it must satisfy the inequality $q^*(l^*-1, l^*, l^*) \leq q < q^*(l^*, l^*+1, l^*+1)$, and applying g_1 to this gives $l^* = g_1(q) + 1$. Similarly, m^* must satisfy the inequality $q^*(m^*-1, m^*-1, m^*) \leq q < q^*(m^*, m^*, m^*+1)$, and applying g_2 to this gives $m^* = g_2(q) + 2$. Finally, n^* must satisfy the inequality $q^*(n^*-1, n^*-1, n^*-1) \leq q < q^*(n^*, n^*, n^*)$, and applying g_3 to this gives $n^* = g_3(q) + 1$. Thus $\chi_{\mathrm{SC}}(K_{3,q}) = 2q + 3 + g_1(q) + g_2(q) + g_3(q)$, and this quantity is equal to $2q + 1 + \lfloor \sqrt{12q+4} \rfloor$. To see this, let $r = \lfloor \sqrt{12q+4} \rfloor$ and $s = \lfloor \sqrt{12q} \rfloor$. If $r = s$ it is easy to check that the two quantities are equal by considering the cases $r \equiv 0, 1, 2$ modulo 3 separately. If $r = s + 1$, then $12q + 1$ is a perfect square not divisible by 3, hence we need only check that the two quantities are equal only for $r \equiv 1, 2$ modulo 3, which is easily seen to be true.

□

Chapter 3

The Sum Choice Number of $P_3 \,\square\, P_n$

We now compute the sum choice number of $P_3 \,\square\, P_n$. We first present a number of lemmas that will be used in the proof. Most are applicable to a variety of situations other than the present one. The calculation of the sum choice number is a rather long case analysis.

3.1 The configuration number

Suppose that v is a vertex of some graph. In what follows, if we write v in a place where we would expect a graph, such as in the pair (v, f), then v refers to the graph consisting of the vertex v alone. Let G be a graph. We will say $R_1, \ldots R_n$ is a *partition* of G if each R_i is an induced subgraph of G, their vertex sets are pairwise disjoint, and $\bigcup_{1 \le i \le n} V(R_i) = V(G)$. Furthermore, if S is a subgraph of G, we use S^C to denote $G - S$.

Let G be a graph, and let S and T be disjoint induced subgraphs of G. We define the size function f_S^T by

$$f_S^T(v) = f(v) - \deg_T(v),$$

for each $v \in V(S)$. If $S = T^C$, then we will omit it from the notation and write f^T. Consider a pair (G, f), and let S be an induced subgraph of G. The *configuration number* of the pair (S, f), denoted $\gamma(S, f)$, is the least integer k such that there exists an f-assignment $\mathcal{C}$ on G from which there are exactly k distinct proper $\mathcal{C}_S$-colorings. Clearly, (G, f) is choosable if and only if $\gamma(G, f) > 0$. In a sense, $\gamma(G, f)$ measures how far (G, f) is from not being choosable.

As an example, $\gamma(P_2, f) = 2$ when $f \equiv 2$, as the f-assignment assigning the same list to each vertex has exactly two proper colorings, and one can easily convince oneself that there is no f-assignment which only has one proper coloring (see Figure 3.1a). On the other hand, consider (C_4, f) with $f \equiv 2$ and the f-assignment shown in Figure 3.1b. Let S be the subgraph induced by the two vertices assigned the list 12. These lists show that $\gamma(S, f) \le 1$. This is because any coloring of C_4 assigning color 1 to the top vertex of S and color 2 to the bottom is not proper, meaning any proper coloring of C_4 must assign color 2 to the top vertex of S and color 1 to the bottom. In fact, it is easy to show that f is choosable, so $\gamma(S, f) = 1$. On the other hand, $\gamma(S, f_S) = 2$, as in (a). The difference here between $\gamma(S, f)$ and $\gamma(S, f_S)$ is that the former considers the influence of all the lists of C_4 on the lists of S, whereas the latter is restricted purely to S.

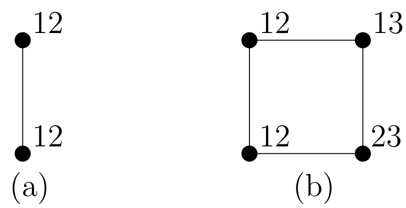


Figure 3.1: Graphs for the first example

A particularly important case is $\gamma(S, f_S) = 1$. In this case, there is some f-assignment from which there is exactly one proper coloring of S. This generalizes the idea of a single vertex receiving list size 1. Recall that Lemma 2.1a states that if $f(v) = 1$, then (G, f) is choosable if and only if $(G - v, f^v)$ is. We will generalize this below (Lemma 3.3) to show that if $\gamma(S, f_S) = 1$, then (G, f) is choosable if and only if (S^C, f^S) is (provided S fits together nicely with S^C). For example, consider $P_3 \,\square\, P_3$ with the size function f shown in Figure 3.2. Let R denote the subgraph induced by the three vertices on the right, and let L denote the subgraph induced by the remaining six vertices. The list assignment given on the right shows $\gamma(R, f_R) \leq 1$, and in fact $\gamma(R, f_R) = 1$, as (R, f_R) is choosable. No proper coloring from these lists can use color 4 or 5 on any of the middle three vertices. Removing colors 4 and 5 from those middle vertices leaves us with familiar lists on L which have no proper coloring. In essence, $(P_3 \,\square\, P_3, f)$ is choosable if and only if (L, g) is choosable where $g \equiv 2$.

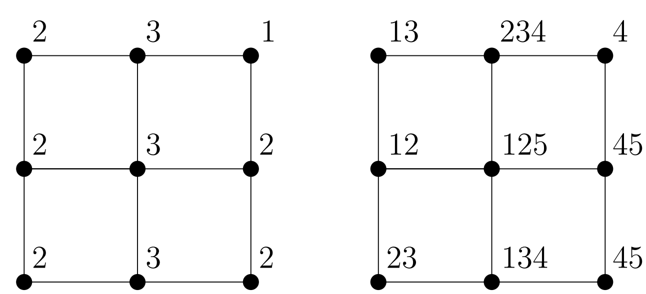


Figure 3.2: An example of Lemma 3.2

Consider now what happens when instead of $\gamma(S, f_S) = 1$, we only have $\gamma(T, f_S) = 1$, for some induced subgraph T of S. For example, consider $P_3 \,\square\, P_4$ with the size function f shown in Figure 3.3. Let S be the subgraph induced by the rightmost two columns, and let T be the subgraph induced by the third column from the left. The f-assignment $\mathcal{C}$ shown on the right is such that every proper $\mathcal{C}_S$-coloring has only one possible restriction to T, namely colors 4, 6, and 5 must be used on the top, middle and bottom vertices, respectively, of T. This shows that $\gamma(T, f_S) \leq 1$, and in fact, one could check that $\gamma(T, f_S) = 1$. Thus, colors 4, 5, and 6 cannot be used on the second column from the left, and so S^C must be colored from familiar lists which have no proper coloring. That is, $\gamma(S^C, f^T_{S^C}) = 0$, and so $\gamma(P_3 \,\square\, P_4, f) = 0$, implying that f is not choosable.

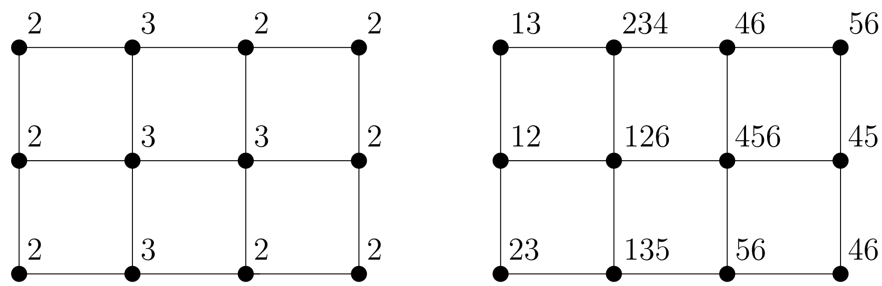


Figure 3.3: A second example of Lemma 3.2

However, given (G, f) with S an induced subgraph of G and T an induced subgraph of S satisfying $\gamma(T, f_S) = 1$, it is not always true that $\gamma(S^C, f) = \gamma(S^C, f^T_{S^C})$. Lemma 3.2 shows, however, that $\gamma(S^C, f) \leq \gamma(S^C, f^T_{S^C})$. As an example, consider $P_3 \,\square\, P_3$ with the size function f shown in Figure 3.4. Let S be the subgraph induced by the right two columns, let T_1 be the subgraph induced by the middle column, and let T_2 be the subgraph induced by the bottom two vertices of the middle column. It is not hard to check that $\gamma(T_1, f_S) = \gamma(T_2, f_S) = 1$. We have $\gamma(S^C, f^{T_1}_{S^C}) = 0$, and so $\gamma(S^C, f) = 0$. However, $\gamma(S^C, f^{T_2}_{S^C}) = 1$, showing that $\gamma(S^C, f) < \gamma(S^C, f^{T_2}_{S^C})$. We lose equality here because there is a subset of S, namely the bottom two rows of the two right columns (call it S'), which satisfies that $\gamma(T_2, f_{S'}) = 1$. That is, for equality to hold, we would need some sort of minimality condition on S with respect to T_2.

On a slightly different note, consider $P_3 \,\square\, P_3$ with the size function f shown in Figure 3.5. Let L denote the subgraph induced by the three left vertices, let M denote the subgraph induced by the three middle vertices, and let R denote the subgraph induced by the three right vertices. The list assignment given on the right shows that $\gamma(M, f_M) \leq 2$, and in fact Lemma 3.6 shows $\gamma(M, f_M) = 2$. The two proper colorings of M use either colors 4, 5, and 4, or colors 5, 4, and 5. If the first coloring is used, then R cannot be properly colored, and if the second coloring is used, then L cannot be properly colored. In summary, we have $\gamma(M, f_M) = 2$, (R, f^M_R) is not choosable, and (L, f^M_L) is not choosable. These facts together imply that $(P_3 \,\square\, P_3, f)$ is not choosable, and more specifically, that $\gamma(M, f) = 0$. We can generalize this idea to show that if $\gamma(M, f_M) = m$, and there are graphs $H_1, \ldots H_p$ attached nicely to M,

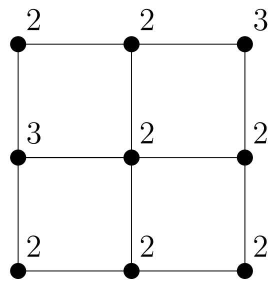


Figure 3.4: A third example of Lemma 3.2

satisfying that $f_{H_i}^M$ is not choosable, then $\gamma(M, f) = \max\{m - p, 0\}$. This is Lemma 3.4.

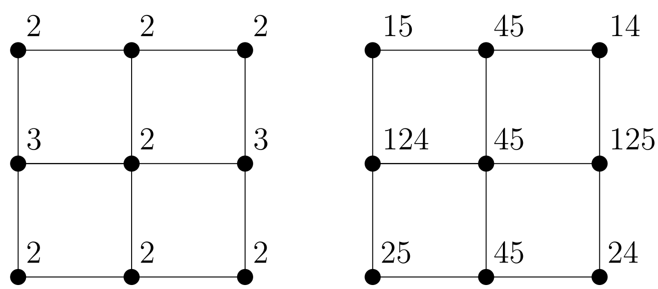


Figure 3.5: An example of Lemma 3.4

3.2 A few general lemmas

We will often implicitly use the contrapositive of the following lemma.

Lemma 3.1. *If (G, f) is choosable, then (S, f_S) is choosable for every induced subgraph S of G.*

Proof. Let $\mathcal{C}'$ be an f_S-assignment. Any extension $\mathcal{C}$ of $\mathcal{C}'$ to all of G has a proper coloring c, by hypothesis. The restriction of c to S is thus a proper $\mathcal{C}$-coloring of S. $\square$

The following lemma is the formal statement of what we mentioned in the second and third examples.

Lemma 3.2. *Let (G, f) be given, and let S be an induced subgraph of G, and T an induced subgraph of S. Suppose that $\gamma(T, f_S) = 1$, and further, suppose that each vertex of S^C is adjacent to at most one vertex of T. Then $\gamma(S^C, f) \le \gamma(S^C, f_{S^C}^T)$. Equality holds if $S = T$.*

This Lemma will be used extensively in the calculation of $\chi_{\mathrm{SC}}(P_3 \square P_n)$. We now prove a more general statement from which Lemma 3.2 follows. The condition on the vertices of S^C in the following lemma means that no vertex of S^C is adjacent to two or more vertices of T that are colored the same by c. This is what we meant earlier when we said the graphs had to "fit together nicely." It is clearly true when each vertex of S^C is adjacent to at most one vertex of T.

Lemma 3.3. *Let (G, f) be given, and let S be an induced subgraph of G. Set $R = S^C$, and let T be an induced subgraph of S. Let $\mathcal{C}$ be an f_S-assignment from which there is exactly one proper $\mathcal{C}_T$-coloring c. Suppose further that for each $v \in V(R)$, $|\{c(w) : w \in V(T), vw \in E(G)\}| = \deg_T(v)$. Then $\gamma(R, f) \leq \gamma(R, f_R^T)$. If, in fact, $S = T$ and $\gamma(T, f_T) = 1$, then $\gamma(R, f) = \gamma(R, f_R^T)$.*

Proof. Let $v_1, \ldots, v_k$ denote the vertices of T, and let $\mathcal{C}'$ be an f_R^T-assignment having exactly $\gamma(R, f_R^T)$ proper colorings. We can name the colors so that $c(v_i) \notin \mathcal{C}'(w)$ for any $v_i \in V(T)$, $w \in V(R)$. Define an f-assignment $\mathcal{D}$ by $\mathcal{D}(v_i) = \mathcal{C}(v_i)$ for $v_i \in V(T)$, and $\mathcal{D}(w) = \mathcal{C}'(w) \cup \{c(v_i) : v_i w \in E(G)\}$ for $w \in V(R)$. By hypothesis this is an f-assignment, and further, $\mathcal{D}_R$ must be colored from $\mathcal{C}'$. Thus, $\gamma(R, f) \leq \gamma(R, f_R^T)$.

Now assume that $S = T$, and let $\mathcal{F}$ be an f-assignment. We will show that there exist at least $\gamma(R, f_R^T)$ proper $\mathcal{F}$-colorings of G, so that $\gamma(R, f) \geq \gamma(R, f_R^T)$. Since $\gamma(T, f_T) = 1$, there exists at least one proper $\mathcal{F}_T$ coloring c' of T. For each $w \in V(R)$, let Q_w denote the set $\{c'(v_i) : v_i w \in E(G)\}$. Let g be a size function on R defined by $g(w) = f(w) - |Q_w|$. Note that $g(w) \geq f_R^T(w)$ for all $w \in V(R)$. Let $\mathcal{F}'$ be the g-assignment given by $\mathcal{F}'(w) = \mathcal{F}(w) \backslash Q_w$. By hypothesis and the fact that $g(w) \geq f_R^T(w)$ for all $w \in V(R)$, there exist at least $\gamma(R, f_R^T)$ proper $\mathcal{F}'$-colorings of R, and hence at least $\gamma(R, f_R^T)$ proper $\mathcal{F}$-colorings of G. $\square$

The following lemma is the formal statement of what we mentioned in the fourth example.

Lemma 3.4. *Let (G, f) be given, and let $R_0, \ldots R_k$ be a partition of G. Set $m = \gamma(R_0, f_{R_0})$, and set $p = \min\{k, m\}$. Let $\mathcal{C}$ be an f_{R_0}-assignment for which there exist exactly m proper $\mathcal{C}$-colorings, $c_1, \ldots, c_m$. Suppose that for each $i = 1, \ldots p$ there exists an index $j(i)$ such that for every $v \in V(R_{j(i)})$, $|\{c_i(w) : vw \in E(G)$ and $w \in V(R_0)\}| = \deg_{R_0}(v)$, and the indices satisfy that if $i_1 \neq i_2$, then $j(i_1) \neq j(i_2)$. Suppose further that $g = f_{R_{j(i)}}^{R_0}$ is not choosable, for each $i = 1, \ldots p$. Then $\gamma(R_0, f) \leq \max\{m - k, 0\}$.*

Proof. We will define an f-assignment $\mathcal{D}$ such that there are at most $\max\{m - k, 0\}$ proper $\mathcal{D}_{R_0}$-colorings. Let $\mathcal{D}(v) = \mathcal{C}(v)$ for $v \in V(R_0)$. By hypothesis, for each $i = 1, \ldots p$ there exists a g-assignment $\mathcal{D}_i$ which has no proper coloring. We can name the colors so that $c_i(w) \notin \mathcal{D}_{j(i)}(v)$ for any $w \in V(R_0)$ and $v \in V(R_{j(i)})$. Now for each $i = 1, \ldots, p$, and for every $v \in V(R_{j(i)})$ define $\mathcal{D}(v)$ to be $\mathcal{D}_{j(i)}(v) \cup \{c_i(w) : vw \in E(G)$ and $w \in V(R_0)\}$. Any $\mathcal{D}$-coloring d which restricts to c_i on R_0, for some

$1 \le i \le p$ cannot be proper as then $R_{j(i)}$ would have to be properly colored from $\mathcal{D}_{j(i)}$. Hence, there are at most $\max\{m-k, 0\}$ proper $\mathcal{D}_{R_0}$-colorings. $\square$

There are two cases where we will typically use this result. The first is with $k = m = 2$, as in the example given earlier. The second is with $k = 1$ and $m = 1$ or 2. When $m = 1$ and $k > 0$, the lemma shows in fact that (G, f) is not choosable. We now give one more technical lemma dealing with configuration numbers.

Lemma 3.5. *Let (G, f) be given. Let S be an induced subgraph of G. Let T and T' be induced subgraphs of S and S^C, respectively, and define $T = T \cup T'$. Suppose $\gamma(T, f_S) \le m$ and $\gamma(T', f_{S^C}) \le m'$. Then $\gamma(T, f) \le \min\{m, m'\}$*

Proof. Let $\mathcal{C}$ be an f_S-assignment having exactly m proper colorings, and let $\mathcal{C}'$ be an f_{S^C}-assignment having exactly m' proper colorings, with the colors named so that $\mathcal{C}(v) \cap \mathcal{C}'(w) = \emptyset$ for any $v \in V(S)$ and $w \in V(S^C)$. Clearly the f-assignment $\mathcal{D}$ given by $\mathcal{D}(v) = \mathcal{C}(v)$ for $v \in V(S)$, and $\mathcal{C}(w) = \mathcal{C}'(w)$ for $w \in V(S^C)$ has exactly $\min\{m, m'\}$ proper colorings. $\square$

Lemma 3.6. *Let f be a choice function on P_n. If* $\text{size}(f) = 2n-1$*, then $\gamma(P_n, f) = 1$, and if* $\text{size}(f) = 2n$*, then $\gamma(P_n, f) = 2$.*

Proof. Denote the vertices of P_n by $v_1, \ldots v_n$ with v_i adjacent to v_{i+1}, for $i = 1, \ldots, n-1$, and let S denote the subgraph induced by the vertices $v_1, \ldots v_{n-1}$. The proof of each statement is by induction on n. The base case $n = 1$ is easy for both statements. Assume that $\gamma(P_{n-1}, g) = 1$ for any choice function g of size $2n-3$, and $\gamma(P_{n-1}, h) = 2$ for any choice function h of size $2n-2$. Let f be a choice function on P_n. If $f(v_n) = 1$, then by Lemma 3.2, $\gamma(P_n, f) = \gamma(S, f^{v_n})$, and by the induction hypothesis this is equal to 1 if $\text{size}(f) = 2n-1$, and 2 if $\text{size}(f) = 2n$. If $\text{size}(f_S) = 2n-3$, then by Lemma 3.2, $\gamma(P_n, f) = \gamma(v_n, f^S) = f^S(v_n)$, and this is equal to 1 if $\text{size}(f) = 2n-1$, and 2 if $\text{size}(f) = 2n$.

Recall that P_n is sc-greedy with sum choice number $2n-1$, so the only other case to consider is when $\text{size}(f) = 2n$ and $f(v_n) = 2$. First, we can construct an f-assignment having exactly two proper colorings as follows. By the induction hypothesis there exists an f_S-assignment $\mathcal{C}'$ having exactly two proper colorings c_1 and c_2. We can extend this to an f-assignment $\mathcal{C}$ by defining $\mathcal{C}(v_i) = \mathcal{C}'(v_i)$ for $i < n$, and $\mathcal{C}(v_n) = \{c_1(v_{n-1}), a\}$, where a is any color not equal to $c_1(v_{n-1})$, if $c_1(v_{n-1}) = c_2(v_{n-1})$, and $a = c_2(v_{n-1})$, otherwise. Clearly there exist exactly two proper $\mathcal{C}$-colorings. On the contrary, let $\mathcal{D}$ be any f-assignment, and let d be a proper $\mathcal{D}_S$-coloring. Since $f(v_n) = 2$, there is some color on $\mathcal{D}(v_n)$ not equal to $d(v_{n-1})$. Hence, we can extend d to a proper coloring of P_n. By the induction hypothesis, there exist at least two proper $\mathcal{D}_S$-colorings. Hence, any f-assignment must have at least two proper colorings. $\square$

The proof above demonstrates our approach to determining the sum choice number of $P_3 \,\square\, P_n$. There are several cases that are easily dealt with by Lemma 3.2, and a subtler case that must be handled with more care.

3.3 The calculation of $\chi_{\mathrm{SC}}(P_3 \,\square\, P_n)$

First, we need some notation. We label the vertices of $P_3 \,\square\, P_n$ as in Figure 3.6. Let H_i denote the subgraph induced by the vertices of column i, namely, $v_{1,i}$, $v_{2,i}$, and $v_{3,i}$. Let T_i denote the top two vertices of H_i, the subgraph induced by $v_{1,i}$ and $v_{2,i}$, and let B_i denote the bottom two vertices of H_i, the subgraph induced by $v_{2,i}$ and $v_{3,i}$. Let L_k denote the subgraph induced by the vertices $\{v_{i,j} : j \leq k\}$, that is, all the vertices to the left of and including column k. Let R_k denote the subgraph induced by the vertices $\{v_{i,j} : j \geq k\}$, that is, all the vertices to the right of and including column k. Note that R_k consists of $n - k + 1$ columns in total.

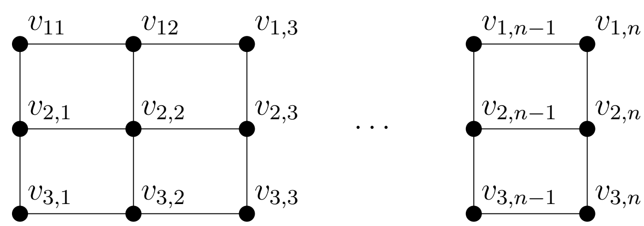


Figure 3.6: $P_3 \,\square\, P_n$

A size function on $P_3 \,\square\, P_n$ is denoted by an array as shown below.

$$\begin{array}{cccc} f(v_{1,1}) & f(v_{1,2}) & \ldots & f(v_{1,n}) \\ f(v_{2,1}) & f(v_{2,2}) & \ldots & f(v_{2,n}) \\ f(v_{3,1}) & f(v_{3,2}) & \ldots & f(v_{3,n}) \end{array}$$

For example, the size function f which assigns 2 to all vertices of $P_3 \,\square\, P_4$ except those of H_4 which get list size 3 is shown below, along with a typical list assignment $\mathcal{C}$ (which is displayed in an analogous way).

$$\begin{array}{cccc} 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 \end{array} \qquad \begin{array}{cccc} 12 & 12 & 23 & 123 \\ 12 & 12 & 23 & 123 \\ 12 & 12 & 34 & 134 \end{array}$$

To indicate f_{H_i} we will use the notation $(f(v_{1,i}), f(v_{2,i}), f(v_{3,i}))$, so that in the example above, $f_{H_4} = (3, 3, 3)$. We will also display list assignments in a similar way, so that in the example above $\mathcal{C}_{H_4} = (123, 123, 134)$. We will write as a shorthand $f_i = f_{H_i}$ and $f^i = f^{H_i}$.

Before proceeding to the theorem, we provide an example to demonstrate the techniques we will use repeatedly in the proof. Refer to the diagram in Figure 3.7. Consider $(P_3 \,\square\, P_2, f)$ with $\mathrm{size}(f_1) = 6$, $f(v_{1,2}) = 1$ and $\mathrm{size}(f_{B_2}) = 6$, and suppose that f is choosable. This is shown in the leftmost "graph" of the diagram. We endeavor to show that $\gamma(H_2, f) = 1$. By Lemma 3.2, it suffices to consider $g = f^{v_{1,2}}$.

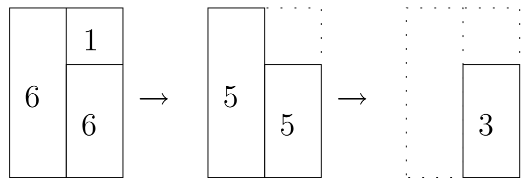


Figure 3.7: An example of a typical reduction in the proof of Theorem 3.7

Notice that $\operatorname{size}(g_1) = 5$ and $\operatorname{size}(g_{B_2}) = 5$. This is indicated in the middle graph of the diagram. The arrow indicates the use of Lemma 3.2. Since $\operatorname{size}(g_1) = 5$, by Lemma 3.6, $\gamma(H_1, g_1) = 1$, and hence by Lemma 3.2, it suffices to consider $h = g^{H_1}$. Notice that $\operatorname{size}(h_{B_2}) = 3$. This is indicated by the right graph of the diagram, and again, the arrow indicates a use of Lemma 3.2. Since $\operatorname{size}(h_{B_2}) = 3$, by Lemma 3.6 we have $\gamma(h_{B_2}, B_2) = 1$. Thus, it follows from Lemma 3.5 that $\gamma(H_2, f) = 1$. The proof contains a large number of reductions of this sort, so for the sake of brevity, we will omit reference to Lemmas 3.2, 3.5, and 3.6. Diagrams such as Figure 3.7 are provided to aid understanding, and in fact, convey the ideas better than the text of the proof.

Theorem 3.7. *The sum choice number of $P_3 \,\square\, P_n$ is* $\mathrm{GB} - \lfloor n/3 \rfloor$. *Explicitly, it is* $8n - \lfloor n/3 \rfloor - 3$.

Proof. We first prove a few claims.

Claim 1. *Suppose that $\gamma(T_{n-1}, f_{L_{n-1}}) = 1$ and $\gamma(B_{n-1}, f_{L_{n-1}}) = 1$. If $f(v_{i,n}) = 1$ for $i = 1$, 2, or 3, or $f(v_{i,n}) = f(v_{i+1,n}) = 2$ for $i = 1$ or 2, then f is not choosable.*

Proof. It suffices to show one of $g = f^{T_{n-1}}$ and $h = f^{B_{n-1}}$ is not choosable, and indeed this is true as $g(v_{i,n}) = 0$ if $f(v_{i,n}) = 1$ for $i = 1$ or 2, $h(v_{3,n}) = 0$ if $f(v_{3,n}) = 0$, and further if $f(v_{1,n}) = f(v_{2,n}) = 2$, then $g(v_{1,n}) = g(v_{2,n}) = 1$, and if $f(v_{2,n}) = f(v_{3,n}) = 2$, then $g(v_{2,n}) = g(v_{3,n}) = 1$. $\square$

In Claim 2 we state the consequences of the preceding lemmas that are used repeatedly throughout the proof. Moreover, we will use Lemma 3.6 implicitly when $n \le 3$. In particular, $\gamma(P_3, f) = 2$ when $\operatorname{size}(f) = 6$, $\gamma(P_3, f) = 1$ when $\operatorname{size}(f) = 5$, and because P_3 is sc-greedy, $\gamma(P_3, f) = 0$ when $\operatorname{size}(f) \le 4$.

Claim 2. *Let f be a size function on $P_3 \,\square\, P_n$, and let g be a choice function on $P_3 \,\square\, P_n$. Let $j \in \mathbb{N}$ such that $1 \le j \le n - 1$.*

(a) If $\gamma(H_j, f_{L_j}) = 1$ and $\operatorname{size}(f_{R_{j+1}}) \le \chi_{\mathrm{SC}}(R_{j+1}) + 2$, then f is not choosable. If $\gamma(T_j, f_{L_j}) = 1$ or $\gamma(B_j, f_{L_j}) = 1$, and $\operatorname{size}(f_{R_{j+1}}) \le \chi_{\mathrm{SC}}(R_{j+1}) + 1$, then f is not choosable.

(b) *If* $\gamma(H_{j+1}, f_{R_{j+1}}) = 1$ *and* $\text{size}(f_{L_j}) \leq \chi_{\text{SC}}(L_j) + 2$, *then* f *is not choosable. If* $\gamma(T_{j+1}, f_{R_{j+1}}) = 1$ *or* $\gamma(B_{j+1}, f_{R_{j+1}}) = 1$, *and* $\text{size}(f_{L_j}) \leq \chi_{\text{SC}}(L_j) + 1$, *then* f *is not choosable.*

(c) *If* $\gamma(H_{n-1}, g_{L_{n-1}}) = 1$ *and* $\text{size}(g_n) = 8$, *then* $\gamma(H_n, g) = 1$. *If* $\gamma(T_{n-1}, g_{L_{n-1}}) = 1$ *or* $\gamma(B_{n-1}, g_{L_{n-1}}) = 1$, *and* $\text{size}(g_n) = 7$, *then* $\gamma(H_n, g) = 1$.

(d) *If* $\text{size}(g_n) = 6$, *and* $\text{size}(g_{L_{n-1}}) \leq \chi_{\text{SC}}(L_{n-1}) + 2$, *then* $\gamma(H_n, g) = 1$. *If* $\text{size}\, g_{T_n} = 4$, *and* $\text{size}(g_{L_{n-1}}) \leq \chi_{\text{SC}}(L_{n-1}) + 1$, *then* $\gamma(T_n, g) = 1$. *The same statement holds with* T_n *replaced by* B_n.

(e) *If* $\text{size}(g_n) = 5$, *then* $\gamma(H_n, g) = 1$.

Proof. Statement (e) follows directly from Lemma 3.6. For each of (a) through (d) we will prove only the first statement. Essentially the same proof works for the second statement. For (a), since $\gamma(H_j, f_{L_j}) = 1$, it suffices to consider $h = f^j_{R_{j+1}}$. However, $\text{size}(h) < \chi_{\text{SC}}(R_{j+1})$, so g is not choosable. The proof of (b) is similar to the proof of (a). For (c), since $\gamma(H_{n-1}, g_{L_{n-1}}) = 1$, it suffices to consider $h = g^{n-1}_n$. However, $\text{size}(h) = 5$, so $\gamma(H_n, h) = 1$. For (d), $\text{size}(g^n_{L_{n-1}}) < \chi_{\text{SC}}(L_{n-1})$, so $g^n_{L_{n-1}}$ is not choosable. Thus, by Lemma 3.4, $\gamma(H_n, g) = 1$. □

Claim 3. *Let* f *be a minimum choice function on* $P_3 \square P_2$. *Then* $\gamma(T_i, f) = \gamma(B_i, f) = 1$, *for* $i = 1, 2$.

Proof. By symmetry, it suffices to prove the result for (T_2, f) and (B_2, f). Recall that $\chi_{\text{SC}}(P_3 \square P_2) = 13$, by Theorem 2.5. We have four cases to consider, $\text{size}(f_2) = 5$, 6, 7, and 8. Note that these are the only cases to be considered, since for (G, f) to be choosable, we require both $\text{size}(f_1)$ and $\text{size}(f_2)$ to be at least 5. The cases $\text{size}(f_2) = 5$, 6, and 8 are immediately taken care of by Claim 2 parts (e), (d), and (c), respectively. For $\text{size}(f_2) = 7$, we have a few cases to consider.

First suppose $f(v_{i,1}) = 1$ for some $i = 1$, 2, or 3. We will show that $\gamma(H_2, f) = 1$. Set $g = f^{v_{i,1}}$ and $H = H_1 - v_{i,1}$. It suffices to show that $\gamma(H_2, g) = 1$. Since $\text{size}(g_2) = 6$, we have $\gamma(H_2, g_2) = 2$. Further, $\text{size}(g^2_H) = 1$ if $i = 2$ and 2 otherwise, so (H, g^2_H) is not choosable. Thus, by Lemma 3.4, $\gamma(H_2, g_2) = 1$ (see Figure 3.8).

Figure 3.8: The two possibilities where $f(v_{i,1}) = 1$

Next suppose $f(v_{i,2}) = 1$ for some $i = 1$, 2, or 3. We will again show that $\gamma(H_2, f) = 1$. Set $g = f^{v_{i,2}}$, and set $H = G - v_{i,2}$. It suffices to show that $\gamma(H, g) = 1$. Note that $\text{size}(g_1) = 5$, so $\gamma(H_1, g_1) = 1$, and thus it suffices to show that $\gamma(H, g^1_H) =$

Figure 3.9: The two possibilities where $f(v_{i,2}) = 1$

1. This is easily seen to be true as size g_H^1 is equal to 2 if $i = 2$ and 3 otherwise (see Figure 3.9).

Thus, it remains to consider $f_1 = (2, 2, 2)$, and $f_2 = (2, 2, 3)$, or $(2, 3, 2)$. By symmetry we need not consider $f_2 = (3, 2, 2)$. Consider first $f_2 = (2, 2, 3)$. By Claim 2(e), $\gamma(T_2, f) = 1$. Now consider the f-assignment $\begin{smallmatrix}13 & 23\\ 12 & 12\\ 23 & 123\end{smallmatrix}$. Any proper coloring from these lists must use colors 2 and 1 on $v_{2,1}$ and $v_{2,2}$, respectively. Thus, color 3 must be used on $v_{3,1}$, and hence color 2 must be used on $v_{3,2}$. These lists thus show that $\gamma(B_2, f) = 1$. Finally, consider $f_2 = (2, 3, 2)$, and the f-assignment $\begin{smallmatrix}13 & 23\\ 12 & 123\\ 23 & 13\end{smallmatrix}$. By assuming color 1 is used on $v_{2,2}$ and tracing through the possibilities, one can easily conclude that no proper coloring uses color 1 on $v_{2,2}$, and similarly for color 2. Hence, color 3 must be used on $v_{2,2}$, which then implies that colors 2 and 1 must be used on $v_{1,2}$ and $v_{3,2}$, respectively. Thus, $\gamma(H_2, f) = 1$. □

Claim 4. *Let f be a choice function on $G = P_3 \,\square\, P_3 - v_{1,1}$. Suppose $f_{B_1} \equiv 2$ and* $\text{size}(f_2) = \text{size}(f_3) = 7$. *Let $\mathcal{C}'$ be a f_{B_1} assignment with $|\mathcal{C}'(v_{2,1}) \cap \mathcal{C}'(v_{3,1})| = 1$. Then there exists an f-assignment whose restriction to B_1 equals $\mathcal{C}'$, for which there is exactly one possible restriction of any proper coloring to T_n, and an f-assignment whose restriction to B_1 equals $\mathcal{C}'$, for which there is exactly one possible restriction of any proper coloring to B_n.*

Proof. We may assume that $\mathcal{C}'(v_{2,1}) = 13$ and $\mathcal{C}'(v_{3,1}) = 23$. Suppose first that $f(v_{1,2}) = 1$. Set $g = f^{v_{1,2}}$. Note that $\text{size}(g_3) = 6$, so $\gamma(H_3, g) = 2$. Set $h = g^3_{B_2}$. By Lemma 3.4, if we can show that there exists an h-assignment agreeing with $\mathcal{C}'$ on B_1, and having no proper coloring, then we may conclude $\gamma(H_3, f) = 1$. Since $\text{size}(h_{B_2}) = 3$, there exists an h_{B_2}-assignment having exactly one proper coloring c, and we may name the colors so that c satisfies $c(v_{i,2}) = i - 1$, for $i = 2, 3$. Combining this with $\mathcal{C}'$ on B_1, we see there is no proper coloring.

Suppose next that $f(v_{2,2}) = 1$. We will define an f-assignment $\mathcal{F}$ such that any proper $\mathcal{F}$-coloring has exactly one possible restriction to H_3. Let c be a proper $\mathcal{F}$-coloring. Define $\mathcal{F}(v_{2,2}) = 1$, and let $\mathcal{F}_{B_1} = \mathcal{C}'$. Then $c(v_{2,2}) = 1$, $c(v_{2,1}) = 3$, and $c(v_{3,1}) = 2$. Let H be the subgraph induced by the vertices $v_{1,2}$, $v_{3,2}$, and the vertices of H_3. Notice that H is a path on five vertices, and as shown above, $\gamma(H^C, f_{H^C}) = 1$. Let $g = f^{H^C}$. It suffices to show that $\gamma(H_3, g) = 1$. However, since $\text{size}(g) = 9$, this follows by Lemma 3.6.

Suppose now that $f(v_{3,2}) = 1$. We will define an f-assignment $\mathcal{F}$ such that any proper $\mathcal{F}$-coloring has exactly one possible restriction to H_3. Let c be a proper $\mathcal{F}$-

coloring. Define $\mathcal{F}(v_{3,2}) = 2$, and let $\mathcal{F}_{B_1} = \mathcal{C}'$. Then $c(v_{3,2}) = 2$, $c(v_{3,1}) = 3$, and $c(v_{2,1}) = 1$. Let $H = T_2 \cup H_3$. As shown above $\gamma(H, f_H) = 1$. Let $g = f^{H^C}$. It suffices to show that $\gamma(H_3, g) = 1$. This follows from Lemma 3.4 since $\mathrm{size}(g_3) = 6$ and $\mathrm{size}(g_{T_2}) = 4$, so $\gamma(H_3, g_3) = 2$ and $g^3_{T_2}$ is not choosable.

Next suppose that $f(v_{i,3}) = 1$ for some $i = 1$, 2, or 3. Set $g = f^{v_{i,3}}$. Then $\mathrm{size}(g_{H_2}) = 6$, so $\gamma(H_2, g) = 2$. Thus, there exists a g_2-assignment $\mathcal{F}'$ having exactly two proper colorings c_1 and c_2, and further we may assume the colors are named so that $c_1(v_{2,2}) = i - 1$ for $i = 2, 3$. Extend $\mathcal{F}'$ to a g-assignment by defining $\mathcal{F} = \mathcal{F}'$ on H_2, $\mathcal{F} = \mathcal{C}'$ on B_1, and letting $\mathcal{F}(v_{i,3}) = \mathcal{F}''(v_{i,3}) \cup c_2(v_{i,2})$ for $i = 1, 2, 3$, where $\mathcal{F}''$ is a $g^{H_2}_{H_3}$-assignment having exactly one proper coloring, with colors named so that $c_2(v_{i,2}) \notin \mathcal{F}(v_{i,3})$ for $i = 1, 2, 3$. Note that $\mathcal{F}''$ exists as $\mathrm{size}(g^2_3) = 5$. Then any proper $\mathcal{F}$-coloring must color H_3 from $\mathcal{F}''$.

The possibilities remaining for f_2 are $(3, 2, 2)$, $(2, 3, 2)$, and $(2, 2, 3)$. For the first case, let $\mathcal{F}'$ be a $f^{B_2}_3$-assignment having exactly one proper coloring. We may name the colors so that $2 \notin \mathcal{F}'(v_{2,3})$ and $1 \notin \mathcal{F}'(v_{3,3})$. Define an f-assignment $\mathcal{F}$ by $\mathcal{F}_{B_1} = \mathcal{C}'$, $\mathcal{F}(v_{2,2}) = 12$, $\mathcal{F}(v_{3,2}) = 12$, $\mathcal{F}(v_{1,3}) = \mathcal{F}'(v_{1,3})$, $\mathcal{F}(v_{2,3}) = \mathcal{F}'(v_{2,3}) \cup \{2\}$, and $\mathcal{F}(v_{3,3}) = \mathcal{F}'(v_{3,3}) \cup \{1\}$. Then any proper $\mathcal{F}$-coloring c must satisfy $c(v_{2,2}) = 2$ and $c(v_{3,2}) = 1$, and thus H_3 must be colored from $\mathcal{F}'$. For the other two possibilities consider the following $f_{B_1 \cup H_2}$-assignments: $\begin{smallmatrix} & 13 \\ 13 & 123 \\ 23 & 12 \end{smallmatrix}$, and $\begin{smallmatrix} & 23 \\ 13 & 12 \\ 23 & 123 \end{smallmatrix}$. For the first case no proper coloring can use color 1 on $v_{2,2}$, so H_2 must be colored from $(13, 23, 12)$. In the second case no proper coloring can use color 2 on $v_{3,2}$, so H_2 must be colored from $(23, 12, 13)$. Notice that these list assignments are permutations of each other. Thus, it suffices to consider the list assignment $\mathcal{E} = (23, 12, 13)$ on H_2, and to look at the possibilities for list assignments on H_3.

The list assignment $(12, 123, 23)$ on H_3 combined with $\mathcal{E}$ is such that any proper coloring c must satisfy $c(v_{1,3}) = 1$, $c(v_{2,3}) = 3$, and $c(v_{3,3}) = 2$. The list assignment $(123, 12, 23)$ on H_3 combined with $\mathcal{E}$ is such that any proper coloring c must satisfy $c(v_{1,3}) = 2$ and $c(v_{2,3}) = 1$. The list assignment $(xxx, 23, 23)$ on H_3 combined with $\mathcal{E}$ is such that any proper coloring c must satisfy $c(v_{2,3}) = 3$ and $c(v_{3,3}) = 2$. The list assignment $(13, 12, 123)$ on H_3 combined with $\mathcal{E}$ is such that any proper coloring c must satisfy $c(v_{2,3}) = 2$ and $c(v_{3,3}) = 1$. The list assignment $(13, 13, xxx)$ on H_3 combined with $\mathcal{E}$ is such that any proper coloring c must satisfy $c(v_{1,3}) = 1$ and $c(v_{2,3}) = 3$. An xxx indicates that the value of that list is irrelevant.

23	13	23	123	23	xxx	23	13	23	13
12	123	12	12	12	23	12	12	12	13
13	23	13	23	13	23	13	123	13	xxx

$\square$

Claim 5. *Let f be a choice function on $P_3 \,\square\, P_n$. Suppose that $\gamma(T_{n-1}, g) = 1$ and $\gamma(B_{n-1}, g) = 1$ for any minimum choice function g on L_{n-1}.*

(a) Suppose $\mathrm{size}(f_{L_{n-1}}) = \chi_{\mathrm{SC}}(L_{n-1}) + 1$ *and* $\mathrm{size}(f_n) = 7$*. If* $f_n \neq (2,3,2)$*, then at least one of* $\gamma(T_n, f)$ *and* $\gamma(B_n, f)$ *equals 1. On the other hand, if* $f_n = (2,3,2)$*, then any* f_n*-assignment of the form* $\mathcal{C}' = (1a_1, 13a_2, 3a_3)$*, can be extended to an* f*-assignment* $\mathcal{C}$ *satisfying that any proper* $\mathcal{C}$*-coloring* c *must have* $c(v_{1,n}) = a_1$*. By symmetry, the same statement holds if we replace* $v_{1,n}$ *by* $v_{3,n}$*.*

(b) Suppose $\mathrm{size}(f_{L_{n-1}}) = \chi_{\mathrm{SC}}(L_{n-1})$ *and* $\mathrm{size}(f_n) = 8$*. If* $f_n \neq (2,4,2)$*, then at least one of* $\gamma(T_n, f)$ *and* $\gamma(B_n, f)$ *equals 1.*

Proof. We will consider statement (a) first. To start, suppose $f(v_{i,n}) = 1$ for some $i = 1$, 2, or 3. Set $g = f^{v_{i,n}}$. Suppose first that $i = 1$. Then $\mathrm{size}(g_{L_{n-1}}) = \chi_{\mathrm{SC}}(L_{n-1})$, so by hypotheses, $\gamma(B_{n-1}, g_{L_{n-1}}) = 1$. We have $\mathrm{size}(g^{B_{n-1}}_{B_n}) = 3$, so $\gamma(B_n, g^{B_{n-1}}_{B_n}) = 1$, and hence $\gamma(H_n, f) = 1$ (see Figure 3.10a). If $i = 3$, the same argument works with B_{n-1} and B_n replaced by T_{n-1} and T_n, respectively. If $i = 2$, then there are three (choosable) possibilities for f_n, $(2,1,4)$, $(4,1,2)$, and $(3,1,3)$. However, $(2,1,4)$ is not possible, as then $\gamma(T_n, f_{T_n}) = 1$, and hence $(L_{n-1}, f^{T_n}_{L_{n-1}})$ is not choosable (see Figure 3.10b). Similarly, $(4,1,2)$ is not possible. If $f_n = (3,1,3)$, then $g(v_{1,n}) = g(v_{3,n}) = 2$, and $\mathrm{size}(g_{L_{n-1}}) = \chi_{\mathrm{SC}}(L_{n-1})$. Thus, by hypothesis $\gamma(B_{n-1}, f_{B_{n-1}}) = 1$. Thus, $\gamma(B_n, f) = 1$ as $g^{B_{n-1}}_{B_n}(v_{3,n}) = 1$ (see Figure 3.10c).

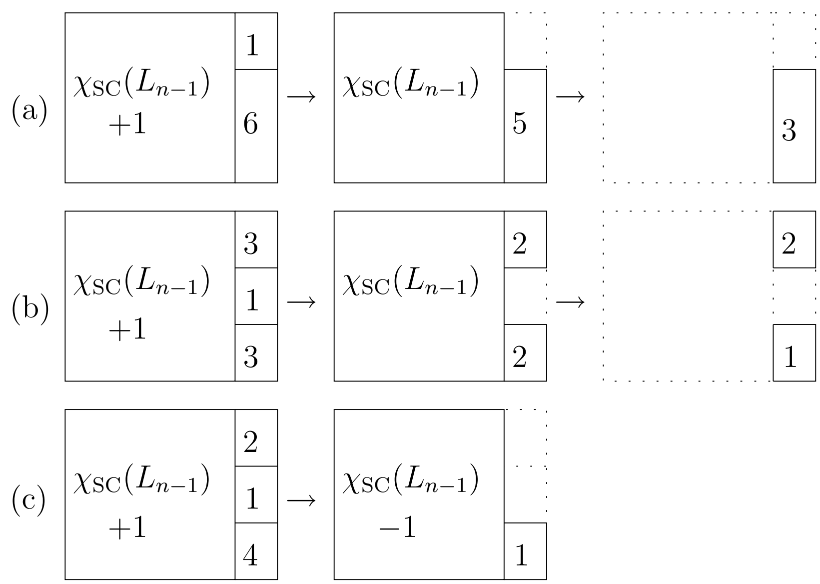


Figure 3.10: The possibilities for Claim 5 part (a)

If $f_n = (2,2,3)$, then $\gamma(T_n, f_{T_n}) = 2$, and moreover, $\mathrm{size}(f^{T_n}_{L_{n-1}}) < \chi_{\mathrm{SC}}(L_{n-1})$. Hence, $\gamma(T_n, f) = 1$ by Lemma 3.4. If $f_n = (3,2,2)$, then the same argument works with T_n replaced by B_n. The only other case is $f_n = (2,3,2)$. Let g be a size function on L_{n-1} defined by $g(v_{1,n-1}) = f(v_{1,n-1}) - 1$ and $g(v_{i,j}) = f(v_{i,j})$ for all other i and j. Choose a g-assignment $\mathcal{C}''$ such that any proper $\mathcal{C}''$-coloring c'' satisfies $c''(v_{i,n-1}) = a_i$,

$i = 2, 3$. This exists by hypotheis since $\text{size}(g_{L_{n-1}}) = \chi_{\text{SC}}(L_{n-1})$. Define an f-assignment $\mathcal{C}$ by $\mathcal{C}(v_{1,n-1}) = \mathcal{C}''(v_{1,n-1}) \cup \{1\}$, $\mathcal{C}(v_{i,n}) = \mathcal{C}'(v_{i,n})$ for $i = 1, 2, 3$, and $\mathcal{C}(v_{i,j}) = \mathcal{C}''(v_{i,j})$ for any other values of i and j. Suppose c were a proper $\mathcal{C}$-coloring and $c(v_{1,n}) = 1$. Then c must restrict to a proper $\mathcal{C}''$-coloring on L_{n-1} which would imply that $c(v_{2,n-1}) = a_2$ and $c(v_{3,n-1}) = a_3$. However, then $c(v_{2,n}) = c(v_{3,n}) = 3$, which is not possible. So any proper $\mathcal{C}$ coloring c must satisfy $c(v_{1,n}) = a_1$.

To prove (b), by Claim 1, note first that f does not assign list size 1 to any vertex of H_n, nor list size 2 to adjacent vertices of H_n. If one of $f(v_{1,n})$ and $f(v_{2,n})$ is 2 and the other is 3, then $\gamma(T_n, f) = 1$, as $\gamma(T_{n-1}, f_{L_{n-1}}) = 1$ by hypothesis. If one of $f(v_{2,n})$ and $f(v_{3,n})$ is 2 and the other is 3, then $\gamma(B_n, f) = 1$, as $\gamma(B_{n-1}, f_{L_{n-1}}) = 1$ by hypothesis. $\square$

To prove Theorem 3.7, we will first we exhibit choice functions of the desired size, then we will show by (strong) induction on n that

$$\chi_{\text{SC}}(P_3 \square P_n) = \chi_{\text{SC}}(P_3 \square P_{n-1}) + \begin{cases} 7 & \text{if } n \equiv 2 \pmod 3 \\ 8 & \text{otherwise} \end{cases},$$

and moreover that any minimum choice function f on $P_3 \square P_n$ satisfies that if $n \equiv 0$ or $1 \pmod 3$, then $\gamma(H_n, f_n) = 1$, and if $n \equiv 2 \pmod 3$, then $\gamma(T_n, f_{T_n}) = 1$ and $\gamma(B_n, f_{B_n}) = 1$. From this the statement of the theorem follows. Our base cases are $n = 1, 2$. For all that follows assume that f is a size function on $P_3 \square P_n$.

Upper bound We will exhibit choice functions of the desired size. First, $P_1 \square P_n$ is a path, hence is sc-greedy. Moreover by Theorem 2.5, $P_2 \square P_n$ is sc-greedy. We will now show that the size function $g = \begin{smallmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 2 & 2 & 2 \end{smallmatrix}$ on $G = P_3 \square P_3$ is choosable.

Consider first the choice function $h \equiv 2$ on $P_2 \square P_2$. Let $H' = P_2 \square P_2 - v_{2,1}$. We will show $\gamma(H', h_H) = 2$. To see this, let $\mathcal{C}$ be a h-assignment with $\mathcal{C}(v_{1,2}) = ab$, and suppose that any proper $\mathcal{C}$-coloring uses color b on $v_{1,2}$. For this to be true, it must be that $\mathcal{C}$ is given by $\begin{smallmatrix} ac & ab \\ cd & ad \end{smallmatrix}$, for some colors c and d. However, as $c \neq d$, there exist at least two proper colorings from these lists. If we assume, without loss of generality, that $c \neq b$, then one proper coloring uses color a on $v_{1,1}$ and another uses color c. Note that if $c \neq b$, but $d = b$, then there are exactly two proper colorings. Having established this, consider now $P_3 \square P_3$. Let $S = B_1 \cup B_2$, which is isomorphic to $P_2 \square P_2$, and let $S' = S - v_{2,1}$. Note that S'^C is a path on 5 vertices, and further, label the vertices of the path as $w_1 = v_{1,1}$, $w_2 = v_{1,2}$, $w_3 = v_{1,3}$, $w_4 = v_{2,3}$, and $w_5 = v_{3,3}$.

Let $\mathcal{D}$ be a g-assignment. We must find a proper $\mathcal{D}$-coloring. Let s be a proper coloring of $\mathcal{D}_S$. We can extend s to a proper coloring of all of G if we can find a proper coloring from the list assignment $\mathcal{D}'$ on S'^C given by $\mathcal{D}'(w_i) = \mathcal{D}(w_i) \backslash s(v_i)$, where v_i is the vertex of S' adjacent to w_i, if it exists. Moreover, we can find such a coloring unless the lists $\mathcal{D}'$ are given by a, ac, cd, bd, b on the vertices w_1 ,w_2, w_3, w_4, w_5, respectively, for some colors a, b, c, and d. If this is the case, then consider another proper $\mathcal{D}_S$-coloring t such that $s_{S'} \neq t_{S'}$. As above, we can extend t to a proper coloring of all of G if we can find a proper coloring from the list assignment

$\mathcal{D}''$ on S'^C given by $\mathcal{D}''(w_i) = \mathcal{D}(w_i)\backslash\{t(v_i)\}$, where v_i is the vertex of S' adjacent to w_i, if it exists. Moreover, we can find such a coloring unless the lists $\mathcal{D}''$ are given by a', $a'c'$, $c'd'$, $b'd'$, b' on the vertices w_1 ,w_2, w_3, w_4, w_5, respectively, for some colors a', b', c', and d'. However it is not possible for the lists to be simultaneously of this form and of the form a, ac, cd, bd, b on the vertices w_1, w_2, w_3, w_4, w_5. Thus, we can extend t to a proper coloring of G.

Now, for $n \equiv 0 \pmod 3$, we get a choice function f on $P_3 \square P_n$ of size $\chi_{\mathrm{SC}}(L_{n-1})+7$ from a choice function g on $P_3 \square P_{n-3}$ by defining $f(v) = g(v)$ for $v \in V(L_{n-3})$, and defining $f_{R_{n-2}}$ to be $\begin{smallmatrix} 3&3&2\\ 4&2&2\\ 3&2&2 \end{smallmatrix}$. To see that this is choosable, let $\mathcal{C}$ be an f-assignment. Let c' be a proper $\mathcal{C}_{L_{n-3}}$-coloring. Note that $f^{n-3}_{R_{n-2}}$ is equivalent to the choice function of size 20 previously given, and hence there exists there exists a proper coloring c'' of R_{n-2} from the $f^{n-3}_{R_{n-2}}$-assignment $\mathcal{D}$ given by $\mathcal{D}(v_{i,n-2}) = \mathcal{C}(v_{i,n-2})\backslash\{c'(v_{i,n-3})\}$ for $i = 1, 2, 3$, and $\mathcal{D}(v) = \mathcal{C}(v)$ for $v \in V(R_{n-1})$. We can then combine c' and c'' to get a proper $\mathcal{D}$-coloring.

Next, for $n \equiv 1, 2 \pmod 3$, we get a choice function f on $P_3 \square P_n$ of size $\chi_{\mathrm{SC}}(L_{n-1}) + 8$ from a choice function g on $P_3 \square P_{n-1}$ by defining $f(v) = g(v)$ for $v \in V(L_{n-1})$, and defining $f_n = (3, 2, 3)$. To see that this is choosable, let $\mathcal{C}$ be an f-assignment. Let c' be a proper $\mathcal{C}_{L_{n-1}}$-coloring. Note that $f^{n-1}_n = (2, 1, 2)$ and is choosable, so there exists a proper coloring c'' of H_n from the f^{n-1}_n-assignment $\mathcal{D}$ given by $\mathcal{D}(v_{i,n-2}) = \mathcal{C}(v_{i,n})\backslash\{c'(v_{i,n})\}$ for $i = 1, 2, 3$. We can then combine c' and c'' to get a proper $\mathcal{D}$-coloring.

Lower Bound

Case $n = 1$: This follows directly from Lemma 3.6.

Case $n = 2$: This is Claim 3.

Case $n \equiv 0 \pmod 3$:

Lower bound: First we will show that $\chi_{\mathrm{SC}}(P_3 \square P_n) \geq \chi_{\mathrm{SC}}(P_3 \square P_{n-1}) + 7$. Consider a size function f of size $\chi_{\mathrm{SC}}(P_3 \square P_{n-1}) + 6$. There are two possibilities, according to the choosable values of size(f_n) and size$(f_{L_{n-1}})$. These two cases are size$(f_n) = 5$ and 6, which immediately follow from Claim 2 parts (b) and (a), respectively.

Minimum Choice Property: Next we must show that if f is a minimum choice function, then $\gamma(H_n, f) = 1$. We have three cases to consider, according to the choosable values of size(f_n) and size$(f_{L_{n-1}})$. The cases size$(f_n) = 5$ and 6 are immediately taken care of by Claim 2 parts (e) and (d), respectively. If size$(f_n) = 7$, then size$(f_{L_{n-1}}) = \chi_{\mathrm{SC}}(L_{n-1})$. By the induction hypothesis, $\gamma(T_{n-1}, f_{L_{n-1}}) = 1$, and Claim 2(c) applies.

Case $n \equiv 1 \pmod 3$:

Lower bound: First we will show that $\chi_{\mathrm{SC}}(L_n) \geq \chi_{\mathrm{SC}}(L_{n-1}) + 8$. Consider a size function f of size $\chi_{\mathrm{SC}}(L_{n-1})+7$. There are two possibilities, according to the choosable values of size$(f_{R_{n-1}})$ and size$(f_{L_{n-2}})$. If, on the one hand, size$(f_{R_{n-1}}) = 13$, then

$\text{size}(f_{L_{n-2}}) = \chi_{\text{SC}}(L_{n-2}) + 1$, and by Claim 3, we know that $\gamma(T_{n-1}, f_{R_{n-1}}) = 1$. Thus, (L_n, f) is not choosable by Claim 2(b). On the other hand, if $\text{size}(f_{R_{n-1}}) = 14$, then $\text{size}(f_{L_{n-2}}) = \chi_{\text{SC}}(L_{n-2})$. By the induction hypothesis, $\gamma(H_{n-2}, f_{L_{n-2}}) = 1$, so (L_n, f) is not choosable by Claim 2(a).

Minimum Choice Property: Second, we must show that if f is a minimum choice function, then $\gamma(H_n, f) = 1$. We have four cases to consider, according to the choosable values of $\text{size}(f_n)$ and $\text{size}(f_{L_{n-1}})$. The cases $\text{size}(f_n) = 5, 6$, are immediately taken care of by Claim 2 parts (e) and (d), respectively. If $\text{size}(f_n) = 8$, then $\text{size}(f_{L_{n-1}}) = \chi_{\text{SC}}(L_{n-1})$, so by the induction hypothesis $\gamma(H_{n-1}, f_{L_{n-1}}) = 1$, and Claim 2(c) applies. Finally, suppose $\text{size}(f_n) = 7$. We consider the choosable values of $\text{size}(f_{n-1})$. Either $\text{size}(f_{n-1}) = 7$ and $\text{size}(f_{L_{n-2}}) = \chi_{\text{SC}}(L_{n-2}) + 1$, or $\text{size}(f_{n-1}) = 8$ and $\text{size}(f_{L_{n-2}}) = \chi_{\text{SC}}(L_{n-2})$. If at least one of $\gamma(T_{n-1}, f_{L_{n-1}})$ and $\gamma(B_{n-1}, f_{L_{n-1}})$ equals 1, then the result follows from Claim 2(a). Otherwise, by Claim 5 there are two possibilities left to consider: $f_{n-1} = (2, 4, 2)$ or $(2, 3, 2)$

In the first case, $\gamma(T_{n-2}, f_{L_{n-2}}) = 1$ by the induction hypothesis. Let $g = f_{R_2}^{T_{n-2}}$. It suffices to show that $\gamma(H_n, g) = 1$. Set $h = g_{R_2 - v_{1,n-1}}^{v_{1,n-1}}$. Since $g(v_{1,n-1}) = 1$, it suffices to show $\gamma(H_n, h) = 1$. However, $\text{size}(h_n) = 6$ so $\gamma(H_n, h_n) = 2$, and $\text{size}(h_{B_{n-1}}) = 4$, so $h_{B_{n-1}}^n$ is not choosable. Thus, by Lemma 3.4, $\gamma(H_n, h) = 1$, and the result follows (see Figure 3.11).

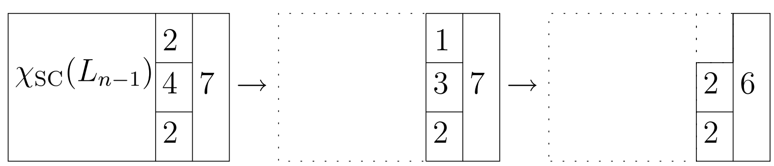
(a) 

Figure 3.11: The possibility $f_{n-1} = (2, 4, 2)$

In the second case, by Claim 5, any f_{n-1}-assignment of the form $(1a_1, 13a_2, a_3)$ can be extended to an $f_{L_{n-1}}$-assignment $\mathcal{C}$ satisfying that any proper $\mathcal{C}$-coloring c must have $c(v_{1,n}) = a_1$, and by symmetry, the same statement holds if we replace $v_{1,n}$ by $v_{3,n}$. Let $\mathcal{D}$ be the extension for the f_{n-1}-assignment $(12, 123, 23)$, so that any proper $\mathcal{D}$-coloring must use color 2 on $v_{1,n-1}$. We consider the possibilities for f_n. If $f(v_{1,n}) = 1$, then extend $\mathcal{D}$ to an f-assignment with no proper coloring by defining $\mathcal{D}(v_{1,n}) = 2$. The values of $\mathcal{D}$ on B_n are irrelevant. By symmetry, if $f(v_{3,n}) = 1$, then f is not choosable. If $f(v_{2,n}) = 1$ and $f(v_{1,n}) = 2$, then extend $\mathcal{D}$ instead by defining $\mathcal{D}(v_{1,n}) = 12$, $\mathcal{D}(v_{2,n}) = 1$. By symmetry, if $f(v_{2,n}) = 1$ and $f(v_{3,n}) = 2$, then f is not choosable. The remaining cases to consider are $f_n = (3, 1, 3)$, $(2, 3, 2)$ and $(2, 2, 3)$. By symmetry, we need not consider $(3, 2, 2)$. The lists for the following cases are shown in Figure 3.12. For the first case, extend $\mathcal{D}$ by defining $\mathcal{D}_{H_n} = (123, 1, 123)$. Let c be a proper $\mathcal{D}$-coloring. Then $c(v_{2,n}) = 1$ and $c(v_{1,n-1}) = 2$ imply that $c(v_{1,n}) = c(v_{2,n-1}) = 3$, and hence $c(v_{3,n-1}) = 2$ and $c(v_{3,n}) = 3$. For the second case, extend $\mathcal{D}$ by defining $\mathcal{D}_{H_n} = (24, 124, 12)$. Let c be a proper $\mathcal{D}$-coloring. Then

$c(v_{1,n-1}) = 2$ implies that $c(v_{1,n-1}) = 4$. Thus, $B_{n-1} \cup B_n$ must be colored from $\begin{smallmatrix}13 & 12\\ 23 & 12\end{smallmatrix}$. Clearly then, we must have $c(v_{2,n}) = 2$ and $c(v_{3,n}) = 1$. For the third case, instead of $\mathcal{D}$, consider the f_{n-1}-assignment $(12, 123, 34)$, which can be extended as above to an $f_{L_{n-1}}$-assignment $\mathcal{F}$ such that any proper $\mathcal{F}$-coloring must use color 2 on $v_{1,n-1}$. Extend $\mathcal{F}$ to all of L_n by defining $\mathcal{F}_{H_n} = (234, 14, 14)$. Let c be a proper $\mathcal{F}$-coloring. Then $c(v_{1,n-1}) = 2$ implies that $B_{n-1} \cup B_n$ must be colored from $\begin{smallmatrix}13 & 14\\ 34 & 14\end{smallmatrix}$. Hence, $c(v_{2,n}) = 4$ and $c(v_{3,n}) = 1$, and therefore, $c(v_{1,n}) = 3$.

12	2	12	12	12	123	12	24	12	234
123		123	1	123	1	123	124	123	14
23		23		23	123	23	12	34	14

Figure 3.12: Lists for the remaining special cases of $n \equiv 1 \pmod 3$

Case $n \equiv 2 \pmod 3$: First, we show that $\chi_{\mathrm{SC}}(P_3 \,\square\, P_n) \geq \chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 8$.

Lower Bound: Let f be a size function of size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_n) + 7$. We consider the choosable values of $\mathrm{size}(f_n)$ and $\mathrm{size}(f_{L_{n-1}})$. First, if $\mathrm{size}(f_n) = 5$, then Claim 2(b) applies. If $\mathrm{size}\, f_n = 7$, then $\mathrm{size}(f_{L_{n-1}}) = \chi_{\mathrm{SC}}(L_{n-1})$, and hence by the induction hypothesis, $\gamma(H_{n-1}, f_{n-1}) = 1$, and Claim 2(a) applies. Thus, any choice function must satisfy $\mathrm{size}(f_n) = 6$. Now consider the choosable values of $\mathrm{size}(f_{R_{n-1}})$ and $\mathrm{size}(f_{L_{n-2}})$. Suppose that $\mathrm{size}(f_{R_{n-1}}) = 15$. Then $\mathrm{size}(f_{L_{n-2}}) = \chi_{\mathrm{SC}}(L_{n-2})$, so by the induction hypothesis, $\gamma(H_{n-2}, f_{L_{n-2}}) = 1$, and Claim 2(a) applies. Thus, it remains to consider $\mathrm{size}(f_{R_{n-1}}) = 13$ or 14.

Suppose that $\mathrm{size}(f_{R_{n-1}}) = 13$. Then both $\gamma(T_{n-1}, f_{R_{n-1}})$ and $\gamma(B_{n-1}, f_{R_{n-1}})$ equal 1 by Claim 3. By Claim 1, no choice function may assign list size 1 to any vertex of H_{n-2}, nor list size 2 to adjacent vertices of H_{n-2}. We conclude that $\mathrm{size}(f_{n-2}) \geq 7$. According to the choosable values of $\mathrm{size}(f_{n-2})$ and $\mathrm{size}(f_{L_{n-3}})$, we then have three possibilities to consider, $\mathrm{size}(f_{n-2}) = 7$, 8, or 9. First, if $\mathrm{size}(f_{n-2}) = 7$, then as $\gamma(T_{n-1}, f_{R_{n-1}}) = 1$ it suffices to consider $g = f^{T_n}_{L_{n-2}}$. However, $\mathrm{size}(g_{n-2}) = 5$, so $\gamma(H_{n-2}, g_{n-2}) = 1$, and thus it suffices to consider $g^{n-2}_{L_{n-3}}$, which is not choosable, as it has size less than $\chi_{\mathrm{SC}}(L_{n-3})$ (see Figure 3.13).

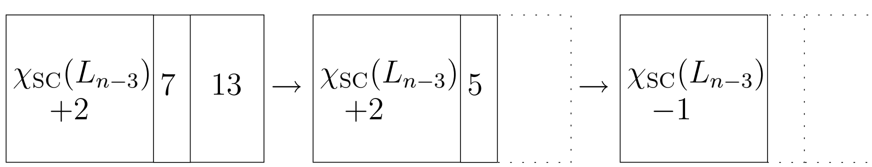


Figure 3.13: The case where $\mathrm{size}(f_{n-2}) = 7$

Next, consider $\mathrm{size}(f_{H_{n-2}}) = 8$. We consider the possible values of $f(v_{1,n-2})$. As mentioned, f does not assign list size 1 to any vertex of H_{n-2}, nor list size 2 to adjacent vertices of H_{n-2}. Thus, if f is choosable, then $2 \leq f(v_{1,n-2}) \leq 3$, for if $f(v_{1,n-2}) \geq 4$, then f would either assign list size 2 to both vertices of B_{n-2} or else list size 1 to one of them. If $f(v_{1,n-2}) = 2$, then it suffices to consider $g = f_{L_{n-2}}^{T_{n-1}}$, since $\gamma(T_{n-1}, f_{R_{n-1}}) = 1$. Then we have $g(v_{1,n-2}) = 1$, so it suffices to consider $h = g_{L_{n-2}-v_{1,n-2}}^{v_{1,n-2}}$. In that case, $\text{size}(h_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3})$, so by the induction hypothesis, $\gamma(B_{n-3}, L_{n-3}) = 1$. Thus, it suffices to consider $h_{B_{n-2}}^{B_{n-3}}$, which is not choosable as it has size 2 (see Figure 3.14).

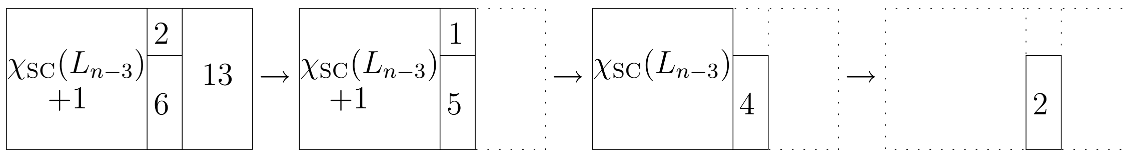


Figure 3.14: The case where $\text{size}(f_{n-2}) = 8$ and $f(v_{1,n-2}) = 2$

If $f(v_{1,n-2}) = 3$, then since $\gamma(B_{n-1}, f_{R_{n-1}}) = 1$, it suffices to consider $g = f_{L_{n-2}}^{B_{n-1}}$. However, $\text{size}(g_{B_{n-2}}) = 3$, so $\gamma(B_{n-2}, g_{B_{n-2}}) = 1$, and thus it suffices to consider $g_{L_{n-3}}^{B_{n-2}}$, which is not choosable, as it has size less than $\chi_{\text{SC}}(L_{n-3})$ (see Figure 3.15). Thus, $f(v_{1,n-2}) \geq 4$, and by symmetry, $f(v_{3,n-2}) \geq 4$. But then $f(v_{2,n-2}) = 0$, so f is not choosable.

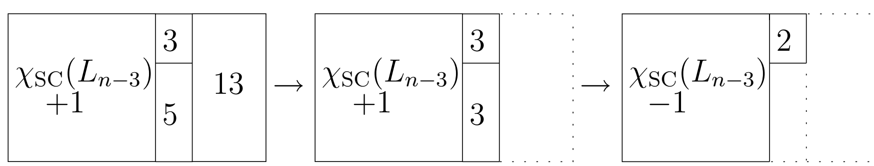


Figure 3.15: The case where $\text{size}(f_{n-2}) = 8$ and $f(v_{1,n-2}) = 3$

Lastly, consider $\text{size}(f_{H_{n-2}}) = 9$. Then $\text{size}(f_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3})$. As mentioned, no choice function assigns list size 1 to any vertex of H_{n-2}. Suppose that $f(v_{1,n-2}) = 2$. By Lemma 3.4, f is not choosable, as $\gamma(v_{1,n-2}, f_{v_{1,n-2}}) = 2$, $\text{size}(f_{L_{n-3}}^{v_{1,n-2}}) < \chi_{\text{SC}}(L_{n-3})$, and $\text{size}(f_{R_{n-1}}^{v_{1,n-2}}) < \chi_{\text{SC}}(R_{n-1})$. Suppose next that $f(v_{3,n-2}) = 3$. Then $\text{size}(f_{B_{n-2}}) = 6$. It suffices to consider $g = f_{L_{n-2}}^{B_{n-1}}$, since $\gamma(B_{n-1}, f_{R_{n-1}}) = 1$. By the induction hypothesis, $\gamma(B_{n-3}, f_{L_{n-3}}) = 1$, so it suffices to consider $h = g_{B_{n-2}}^{n-3}$. However, $\text{size}(h) = 2$, so h is not choosable (see Figure 3.16). Thus, for f to be choosable, $f(v_{1,n-2}) \geq 4$, and by symmetry, $f(v_{3,n-2}) \geq 4$. However, this would then require $f(v_{2,n-2}) = 1$, which is not possible by Claim 1.

Next, we must consider $\text{size}(f_{R_{n-1}}) = 14$. As shown earlier it suffices only to consider $\text{size}(f_n) = 6$. Thus, we must have $\text{size}(f_{n-1}) = 8$. We have four cases to consider according to the choosable values of $\text{size}(f_{n-2})$ and $\text{size}(f_{L_{n-3}})$. If $\text{size}(f_{H_{n-2}}) = 5$, then by the case $n = 3$, $(R_{n-2}, f_{R_{n-2}})$ is not choosable as $\text{size}(f_{R_{n-2}}) = 19 < \chi_{\text{SC}}(R_{n-2})$. If $\text{size}(f_{H_{n-2}}) = 6$, then $\gamma(H_{n-2}, f_{n-2}) = 2$. Furthermore, $\text{size}(f_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3}) + 2$,

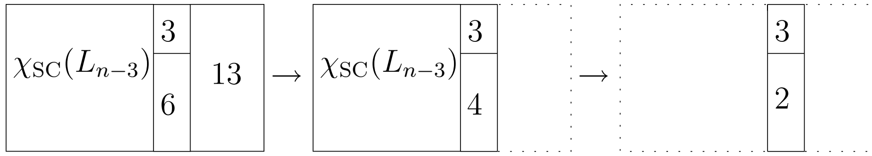


Figure 3.16: The case where size$(f_{n-2}) = 9$ and $f(v_{1,n-2}) = 3$

so $f^{n-2}_{L_{n-3}}$ is not choosable, and size$(f_{R_{n-1}}) = 14$, so $f^{n-2}_{R_{n-1}}$ is not choosable. Thus, by Lemma 3.4, f is not choosable.

If size$(f_{H_{n-2}}) = 8$, then size$(f_{L_{n-3}}) = \chi_{\rm SC}(L_{n-3})$. Note that size$(f_{R_{n-1}}) = 14$, so if $\gamma(T_{n-2}, L_{n-2}) = 1$ or $\gamma(B_{n-2}, L_{n-2}) = 1$, then $f^{T_{n-2}}_{R_{n-1}}$ is not choosable. Thus, by Claim 5(b), it only remains to consider $f_{H_{n-2}} = (2, 4, 2)$. By the induction hypothesis, $\gamma(T_{n-3}, f_{L_{n-3}}) = 1$, so it suffices to consider $g = f^{T_{n-3}}$, and since $g(v_{1,n-2}) = 1$, it suffices to consider $h = g^{v_{1,n-2}}$. Now, size$(h_{R_{n-1}}) = 13$, so by Claim 3, $\gamma(B_{n-1}, h_{B_{n-1}}) = 1$. However, size$(h_{B_{n-2}}) = 4$, so $h^{B_{n-1}}_{B_{n-2}}$ is not choosable (see Figure 3.17).

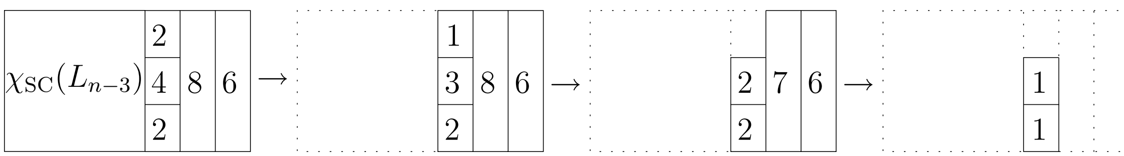


Figure 3.17: The case where size$(f_{R_{n-1}}) = 14$ and $f_{H_{n-2}} = (2, 4, 2)$

Finally, we have to consider size$(f_{H_{n-2}}) = 7$. In this case, size$(f_{L_{n-3}}) = \chi_{\rm SC}(L_{n-3}) + 1$. If $\gamma(T_{n-2}, f_{L_{n-2}}) = 1$, then (L_n, f) is not choosable, as $(R_{n-1}, f^{T_{n-2}}_{R_{n-1}})$ is not choosable. A similar argument holds if $\gamma(B_{n-2}, f_{L_{n-2}}) = 1$. Otherwise, by Claim 5, $f_{n-2} = (2, 3, 2)$ and the f_{n-2}-assignment $(12, 123, 34)$, can be extended to an $f_{L_{n-2}}$-assignment $\mathcal{C}'$ satisfying that any proper $\mathcal{C}'$-coloring c must have $c(v_{1,n}) = 2$. Since size$(f^{v_{1,n-2}}_{R_{n-1}}) = 13$, by Claim 3, there exists an $f^{v_{1,n-2}}_{R_{n-1}}$-assignment $\mathcal{C}''$ such that $2 \notin \mathcal{C}''(v_{1,n-1})$ and any proper $\mathcal{C}''$-coloring c'' must satisfy $c''(v_{2,n-1}) = 1$, $c''(v_{3,n-1}) = 4$ (we can choose the names of the colors as such). Define an f-assignment $\mathcal{C}$ by $\mathcal{C}(v) = \mathcal{C}'(v)$ for $v \in V(L_{n-2})$, $\mathcal{C}(v_{1,n-1}) = \mathcal{C}'(v_{1,n-1}) \cup \{1\}$, and $\mathcal{C}(v) = \mathcal{C}''(v)$ for any other $v \in V(R_{n-1})$. Then any proper $\mathcal{C}$-coloring must have $c(v_{2,n-1}) = 1$ and $c(v_{3,n-1}) = 4$, but in that case, $c(v_{2,n-2}) = c(v_{3,n-2}) = 1$, which is not possible.

Minimum Choice Property: Next we show that any minimum choice function f satisfies $\gamma(T_n, f) = 1$ and $\gamma(B_n, f) = 1$. First, if size$(f_{R_{n-1}}) = 13$, then the result follows from the Claim 3. If size$(f_{R_{n-1}}) = 16$, then size$(f_{L_{n-2}}) = \chi_{\rm SC}(L_{n-2})$. Thus, by the induction hypothesis, $\gamma(H_{n-2}, f_{L_{n-2}}) = 1$, so it suffices to consider $f^{n-2}_{R_{n-1}}$. As size$(f^{n-2}_{R_{n-2}}) = 13$, the result follows from Claim 3. Consider now the choosable values of size(f_n) and size$(f_{L_{n-1}})$. The cases size$(f_n) = 5$ and 6 follow immediately from Claim 2 parts (b) and (d), respectively. If $\text{size}(f_{H_n}) = 8$, then $\text{size}(f_{L_{n-1}}) = \chi_{\text{SC}}(L_{n-1})$, so by the induction hypothesis, Claim 2(c) applies. Thus, it remains to consider the cases $\text{size}(f_{H_n}) = 7$ and $\text{size}(f_{R_{n-1}}) = 14, 15$.

Consider first $\text{size}(f_{R_{n-1}}) = 15$. Then $\text{size}(f_{L_{n-2}}) = \chi_{\text{SC}}(L_{n-2}) + 1$. We consider possibilities according to the choosable values of $\text{size}(f_{n-2})$ and $\text{size}(f_{L_{n-3}})$. If $\text{size}(f_{n-2}) = 5$, then $\gamma(H_{n-2}, f_{n-2}) = 1$, so it suffices to consider $f^{n-2}_{R_{n-1}}$, which is of size 12 and hence is not choosable. If $\text{size}(f_{H_{n-2}}) = 6$, then f is not choosable by Lemma 3.4, as $\gamma(H_{n-2}, f_{n-2}) = 2$, and $(L_{n-3}, f^{n-2}_{L_{n-3}})$ and $(R_{n-1}, f^{n-2}_{R_{n-1}})$ are not choosable. Suppose now that $\text{size}(f_{n-2}) = 7$. In this case $\text{size}(f_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3}) + 1$. If $\gamma(T_{n-2}, f_{L_{n-2}}) = 1$, then it suffices to consider $g = f^{T_{n-2}}_{R_{n-1}}$. However, $\text{size}(g) = 13$, so the result follows from Claim 3. A similar argument holds if $\gamma(B_{n-2}, f_{L_{n-2}}) = 1$. Otherwise, by Claim 5, $f_{H_{n-2}} = (2, 3, 2)$ and the $f_{H_{n-2}}$-assignment $(12, 123, 23)$ can be extended to an $f_{L_{n-2}}$-assignment $\mathcal{C}'$ satisfying that any proper $\mathcal{C}$-coloring c must have $c(v_{1,n}) = 2$. The result then follows from Claim 4 applied to $f^{v_{1,n-2}}_{R_{n-2}}$. Finally, consider $\text{size}(f_{n-2}) = 8$. Then $\text{size}(f_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3})$. Suppose that $f(v_{1,n-2}) = 3$. Then $\text{size}(f_{B_{n-2}}) = 5$. By the induction hypothesis $\gamma(B_{n-3}, f_{L_{n-3}}) = 1$, so it suffices to consider $g = f^{B_{n-3}}_{R_{n-2}}$. Notice then that $\text{size}(g_{B_{n-2}}) = 3$, so $\gamma(B_{n-2}, g_{B_{n-2}}) = 1$. Thus, it suffices to consider $h = g^{B_{n-2}}_{R_{n-1}}$. However, $\text{size}(h_{R_{n-1}}) = 13$, so the result follows from the Claim 3 (see Figure 3.18a). By Claim 1, f does not assign list size 1 to any vertex of H_{n-2}, nor list size two to any two adjacent vertices of H_{n-2}, so it remains to consider $f_{n-2} = (2, 4, 2)$. By the induction hypothesis, $\gamma(T_{n-3}, f_{L_{n-3}}) = 1$, so it suffices to consider $g = f^{T_{n-3}}_{R_{n-2}}$. Note that $g_{n-2} = (1, 3, 2)$. Thus, it suffices to consider $h = g^{v_{1,n-2}}_{R_{n-2} - v_{1,n-2}}$. Notice that $\text{size}(h_{n-1}) = \text{size}(h_n) = 7$ and $h(v_{2,n-2}) = h(v_{3,n-2}) = 2$, so the result follows from Claim 4 (see Figure 3.18b).

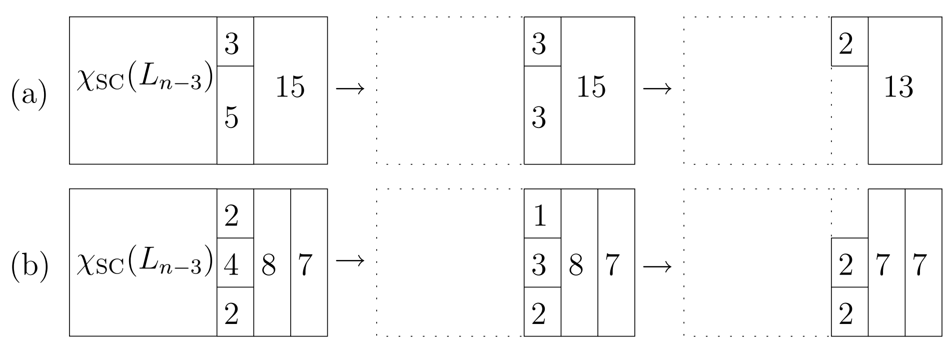


Figure 3.18: Two cases where $\text{size}(f_{R_{n-1}}) = 15$ and $\text{size}(f_{n-2}) = 8$

The final case to consider is $\text{size}(f_{R_{n-1}}) = 14$. From earlier discussion, we only need to consider $\text{size}(f_{H_n}) = \text{size}(f_{H_{n-1}}) = 7$. We consider cases, according to the choosable values of $\text{size}(f_{n-2})$ and $\text{size}(f_{L_{n-3}})$. If $\text{size}(f_{n-2}) = 5$, then $\text{size}(f_{R_{n-2}}) = 19 < \chi_{\text{SC}}(R_{n-2})$, by the case $n = 3$. If $\text{size}(f_{n-2}) = 6$, then $\text{size}(f_{R_{n-2}}) = 20 = \chi_{\text{SC}}(R_{n-2})$, so by the case $n = 3$, $\gamma(H_n, f_{R_{n-2}}) = 1$. Now suppose size $f_{H_{n-2}} = 7$. If f assigns list size 1 to any vertex $v_{i,n-2}$ for any $i = 1$, 2, or 3, then it suffices to consider $f_{R_{n-1}}^{v_{i,n-2}}$, which is of size 13, and hence the result follows from Claim 3. If f assigns list size 2 to two adjacent vertices of H_{n-2}, then the result follows from Claim 4. Otherwise, we must have $f_{n-2} = (2, 3, 2)$. Define a size function g on L_{n-3} by $g(v) = f(v)$ for $v \in V(L_{n-4})$, and $g(v) = f(v) - 1$ for $v \in V(H_{n-3})$. Let $\mathcal{C}'$ be a g-assignment having no proper coloring, which exists because $\text{size}(g) < \chi_{\text{SC}}(L_{n-3})$. Let $\mathcal{C}$ be the $f_{L_{n-2}}$-assignment defined by $\mathcal{C}_{H_{n-2}} = (12, 123, 23)$, $\mathcal{C}(v) = \mathcal{C}'(v)$ for $v \in V(L_{n-4})$, and $\mathcal{C}(v_{i,n-3}) = \mathcal{C}'(v_{i,n-3}) \cup \{i\}$ for $i = 1, 2, 3$. We may name the colors of $\mathcal{C}'$ such that color i does not appear on $\mathcal{C}'(v_{i,n-3})$ for $i = 1$, 2, or 3. Then no proper $\mathcal{C}$-coloring may use color 2 on $v_{2,n-2}$, as otherwise L_{n-4} would have to be properly colored from $\mathcal{C}'$. The result then directly follows from Claim 4.

Next, suppose $\text{size}(f_{H_{n-2}}) = 8$. In this case, $\text{size}(f_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3}) + 1$. Our argument is quite analogous to that of the previous paragraph. As above, the result follows if f assigns list size 1 to any vertex of H_{n-2}, or if f assigns list size 2 to adjacent vertices of H_{n-2}. Suppose that $f(v_{1,n-2}) = 2$ and $f(v_{2,n-2}) = 3$. Define a size function g on L_{n-3} by $g(v) = f(v) - 1$ for $v \in V(T_{n-3})$ and $g(v) = f(v)$ for any other $v \in V(L_{n-3})$. Then $\text{size}(g) < \chi_{\text{SC}}(L_{n-3})$, so there exists a g-assignment $\mathcal{C}'$ having no proper coloring. Define an $f_{L_{n-3} \cup T_{n-2}}$-assignment $\mathcal{C}$ by $\mathcal{C}(v_{1,n-2}) = 12$, $\mathcal{C}(v_{2,n-2}) = 123$, $\mathcal{C}(v_{i,n-3}) = \mathcal{C}(v_{i,n-3}) \cup \{i\}$, for $i = 1, 2$, and $\mathcal{C}(v) = \mathcal{C}'(v)$ for any other $v \in V(L_{n-3})$. We may name the colors of $\mathcal{C}'$ such that color i does not appear on $\mathcal{C}'(v_{i,n-3})$ for $i = 1, 2$. Then no proper $\mathcal{C}$-coloring can use color 2 on $v_{2,n-2}$, as otherwise L_{n-3} would have to be properly colored from $\mathcal{C}'$. We may thus appeal to Claim 4. A similar argument applies if any other adjacent vertices of H_{n-2} are assigned list sizes 2 and 3, respectively. So it suffices to consider $f_{H_{n-2}} = (2, 4, 2)$. For this, we have a number of cases.

First, if $f_{n-1} = (2, 2, 3)$, then $\gamma(T_{n-1}, f_{T_{n-1}}) = 2$. If we can show that $g = f_{L_{n-3}}^{T_{n-1}}$ is not choosable, then it follows from Lemma 3.4 that $\gamma(T_{n-1}, f_{L_{n-1}}) = 1$, and hence it would suffice to consider $f_n^{T_{n-1}}$, which is of size 5, and from there one concludes that $\gamma(H_n, f_n^{T_{n-1}}) = 1$. To show g is not choosable, note that $g_{n-2} = (1, 3, 2)$, so it suffices to consider $h = g^{v_{1,n-2}}$. We have $\text{size}(h_{L_{n-3}}) = \chi_{\text{SC}}(L_{n-3})$, so by the induction hypothesis, $\gamma(B_{n-3}, h_{L_{n-3}}) = 1$. Thus, it suffices to consider $h_{B_{n-2}}^{B_{n-3}}$, which has size 2 and hence is not choosable (see Figure 3.19). By symmetry, $\gamma(H_n, f) = 1$ if $f_{n-1} = (3, 2, 2)$.

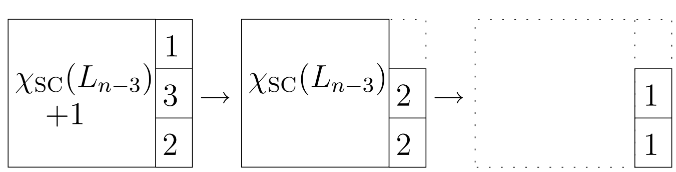


Figure 3.19: Part of the case where $f_{n-1} = (2, 2, 3)$

Next, suppose that $f(v_{1,n-1}) = 1$. Let H be the subgraph induced by $v_{1,n-1}$ and $v_{1,n-2}$. Note that $\gamma(H, f_H) = 1$, so it suffices to consider $g = f^H$. Since $\text{size}(g_{L_{n-3}}) =$

$\chi_{\text{SC}}(L_{n-3})$, by the induction hypothesis $\gamma(B_{n-3}, g_{L_{n-3}}) = 1$, so it suffices to consider $h = g^{B_{n-3}}$. Since $\text{size}(h_{B_{n-2}}) = 3$, we have $\gamma(h_{B_{n-2}}, h_{B_{n-2}}) = 1$, so it suffices to consider $\phi = h^{B_{n-2}}$. Since $\text{size}(\phi_{B_{n-1}}) = 3$, we have $\gamma(\phi_{B_{n-1}}, \phi_{B_{n-1}}) = 1$. However, $\phi_n^{B_{n-1}}$ has size 4, and hence is not choosable (see Figure 3.20). By symmetry, f is not choosable if $f(v_{3,n-1}) = 1$.

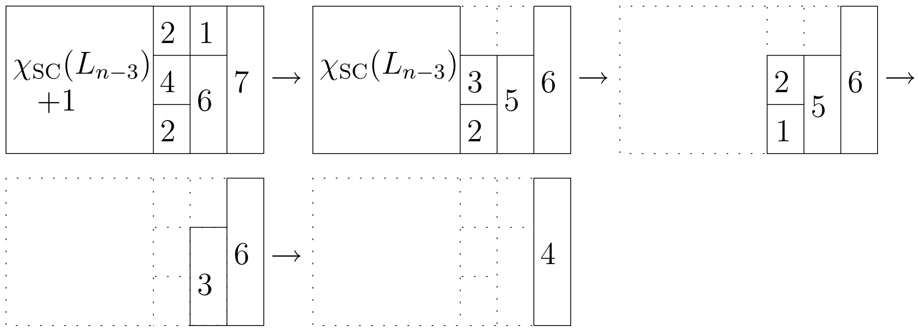


Figure 3.20: The case where $f(v_{1,n-1}) = 1$

Next, suppose that $f(v_{2,n-1}) = 1$. It suffices to consider $g = f^{v_{2,n-1}}$. Note that $g_{H_{n-2}} = (2, 3, 2)$, so by Claim 5, $\gamma(v_{1,n-2}, g_{L_{n-2}}) = 1$. Thus, it suffices to consider $h = g^{v_{1,n-2}}$. Since $h(v_{1,n-1}) + h(v_{3,n-1}) = 3$, either $h(v_{1,n-1}) = 1$ or $h(v_{3,n-1}) = 1$. Thus, either $\text{size}(h_n^{v_{1,n-1}}) = 5$ or $\text{size}(h_n^{v_{3,n-1}}) = 5$. We conclude that $\gamma(H_n, f) = 1$ (see Figure 3.21).

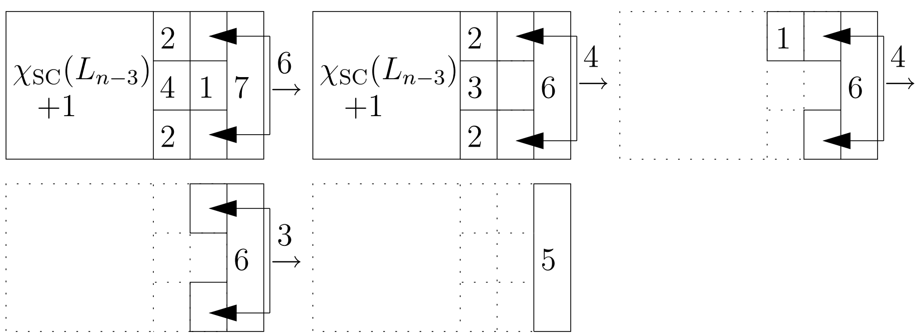


Figure 3.21: The case where $f(v_{2,n-1}) = 1$

Consider now the possibilities for f_n. First suppose that $f(v_{i,n}) = 1$ for some $i = 1$, 2, or 3. We argue analogously as in Claim 3. Set $g = f^{v_{i,n}}$, and set $v_{i,n}^C$. It suffices to show that $\gamma(H, g) = 1$. Note that $\text{size}(g_{L_{n-1}}) = \chi_{\text{SC}}(L_{n-1})$, so $\gamma(H_{n-1}, g_{L_{n-1}}) = 1$. Thus it suffices to show that $\gamma(H_n, g_n^{n-1}) = 1$. This is easily seen to be true, as size g_n^{n-1} is equal to 2 if $i = 2$, and 3 otherwise (see Figure 3.22).

From the above paragraphs, it remains to consider $f_{n-2} = (2, 4, 2)$, $f_{n-1} = (2, 3, 2)$, and $f_n = (2, 2, 3)$ or $(2, 3, 2)$. By symmetry, we need not consider $f_n = (3, 2, 2)$. Define a size function g on R_{n-2} by $g(v_{2,n-2}) = f(v_{2,n-2}) - 1$ and $g(v) = f(v)$ for any other $v \in V(R_{n-2})$. Define a size function h on L_{n-2} by $h(v) = f(v) - 1$ for $v \in V(T_{n-3})$ and $h(v) = f(v)$ for any other $v \in V(L_{n-3})$, and let $\mathcal{C}'$ be an h-assignment having no proper coloring, which exists by the induction hypothesis. We may assume that the colors are named so that color i is not in $\mathcal{C}'(v_{i,n-3})$ for $i = 1, 2$. Define the following g-assignments, $\mathcal{D}_1$ and $\mathcal{D}_2$.

$$\mathcal{D}_1 = \begin{array}{ccc} 12 & 23 & 13 \\ 134 & 123 & 12 \\ 14 & 13 & 23x \end{array} \qquad \mathcal{D}_2 = \begin{array}{ccc} 12 & 23 & 13 \\ 134 & 123 & 123 \\ 14 & 13 & 23 \end{array},$$

where the x can be any color other than 2 or 3. Define an f-assignment $\mathcal{C}$ by $\mathcal{C}(v_{i,n-3}) = \mathcal{C}'(v_{i,n-3}) \cup \{i\}$ for $i = 1, 2$, $\mathcal{C}(v) = \mathcal{C}'(v)$ for any other $v \in V(L_{n-3})$, $\mathcal{C}(v_{2,n-2}) = \mathcal{D}_j(v_{2,n-2}) \cup \{2\}$, and $\mathcal{C}(v) = \mathcal{D}_j(v)$ for any other $v \in V(R_{n-2})$, where $j = 1$ if $f_{H_n} = (2, 2, 3)$ and $j = 2$ if $f_{H_n} = (2, 3, 2)$. Then no proper $\mathcal{C}$-coloring may use color 2 on $v_{2,n-2}$, as otherwise L_{n-3} would have to be properly colored from $\mathcal{C}'$, which is impossible.

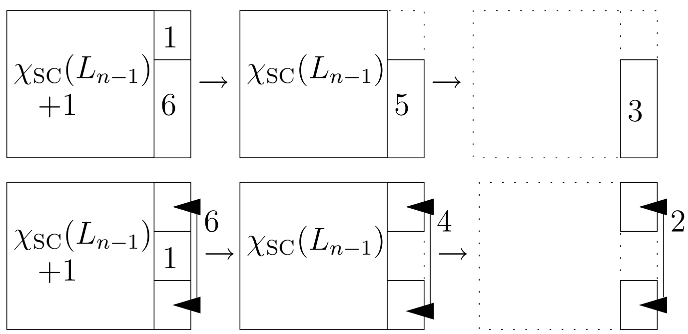


Figure 3.22: The two possibilities where $f(v_{i,n}) = 1$

With the above setup, consider now $f_{H_n} = (2, 2, 3)$. We first note that $\gamma(T_n, f) = 1$ follows directly from Claim 2(b), since $\text{size}(f_{T_n}) = 4$ and $\text{size}(f_{L_{n-1}})\,\chi_{SC}(L_{n-1}) + 1$. Now, we show $\gamma(B_n, f) = 1$. By the above paragraph it suffices to verify that any proper $\mathcal{D}_1$-coloring c must satisfy $c(v_{2,n}) = 2$ and $c(v_{3,n}) = x$. One can easily check this by first supposing that $c(v_{2,n}) = 1$, tracing through to get a contradiction, and then supposing that $c(v_{3,n}) = 3$, tracing through using the fact that $c(v_{2,n}) = 1$ to get a contradiction. Consider next $f_{H_n} = (2, 3, 2)$. We show that $\gamma(H_n, f) = 1$, and by the above paragraph, it suffices only to verify that any proper $\mathcal{D}_2$-coloring must satisfy $c(v_{1,n}) = 1$, $c(v_{2,n}) = 3$, and $c(v_{3,n}) = 2$. One can check this by first supposing that $c(v_{2,n}) = 1$, tracing through to get a contraction, and then supposing that $c(v_{2,n}) = 2$, tracing through again to get a contradiction.

If $\text{size}(f_{n-2}) = 9$, then $\text{size}(f_{L_{n-3}}) = \chi_{SC}(L_{n-3})$, so by the induction hypothesis, $\gamma(T_{n-3}, f_{L_{n-3}})$ and $\gamma(B_{n-3}, f_{L_{n-3}})$ are both 1. Thus, by Claim 1, f cannot assign list size 1 to any vertex of H_{n-2}. Suppose next that $f(v_{1,n-2}) = 2$. Since $\gamma(T_{n-3}, f_{L_{n-3}}) =$

1, it suffices to consider $g = f_{R_{n-2}}^{T_{n-3}}$. However, $g(v_{1,n-2}) = 1$, so it suffices to consider $g_{R_{n-1}}^{v_{1,n-2}}$, which is of size 13. Thus, the result follows from Claim 3 (see Figure 3.23a). A similar argument works if $f(v_{2,n-2}) = 2$ or $f(v_{3,n-2}) = 2$. Thus, it remains only to consider $f_{n-2} = (3, 3, 3)$. By the induction hypothesis, $\gamma(B_{n-3}, f_{L_{n-3}}) = 1$. Thus, it suffices to consider $g = f_{R_{n-2}}^{B_{n-3}}$. Notice that $g(v_{2,n-2}) = g(v_{3,n-2}) = 2$, so the result follows from Claim 4 (see Figure 3.23b).

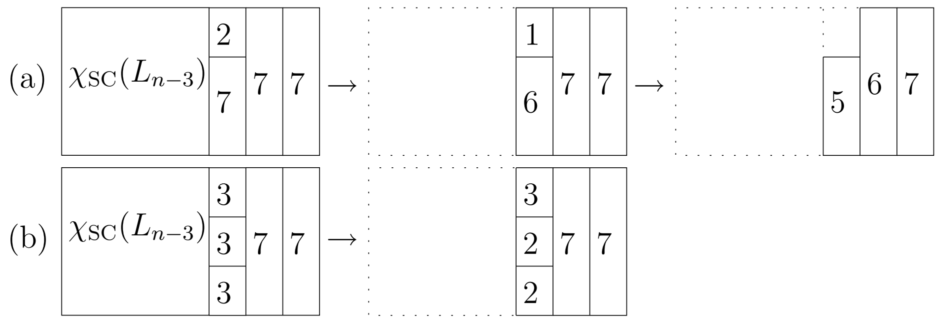


Figure 3.23: The case size$(f_{n-2}) = 9$

□

Here is a table summarizing the minimum choice functions given in the proof. The long entry gives the list sizes by column, that is, size(f_1), size(f_2), ...

n	χ_{SC}	list sizes by column
1	5	5
2	13	7,6
3	20	7,7,6
4	28	7,7,6,8
5	36	7,7,6,8,8
6	43	7,7,6,10,7,6
7	51	7,7,6,10,7,6,8
8	59	7,7,6,10,7,6,8,8
9	66	7,7,6,10,7,6,10,7,6
10	74	7,7,6,10,7,6,10,7,6,8

Chapter 4

Fan Graphs

The *fan graph* F_n is $F_n = P_n \vee K_1$, obtained by joining a vertex to the path P_n. In this chapter we consider the choosability of fan graphs. In particular, we answer a question raised by Isaak, and independently by Pelsmajer and Albertson. They asked if all outerplanar graphs are sc-greedy. We answer this question in the negative by showing that fan graphs are not sc-greedy in general, and that in fact, the gap between the greedy bound and the sum choice number can be arbitrarily large.

We label the vertices of F_n so that the vertices of the path are $v_1, \ldots v_n$, with v_i adjacent to v_{i+1} for $i = 1, \ldots, n-1$, and v_0 is the vertex joined to the path, adjacent to v_i for $i = 1, \ldots, n$ (see Figure 4.1). For simplicity, we will shorten the notation from v_i to i.

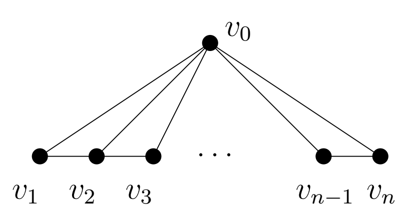


Figure 4.1: The fan graph F_n.

We will concern ourselves with $\tau(F_n)$, defined in Chapter 2 to be the minimum size of a choice function f satisfying $2 \leq f(v) \leq \deg(v)$ for all vertices v. Recall that $\chi_{\text{SC}}(G) = \min\{\tau(G), \rho(G)\}$. Thus, we will be considering size functions g that satisfy $g(1) = g(n) = 2$, $2 \leq f(i) \leq 3$ for $i = 2, \ldots, n-1$, and $2 \leq g(v_0) \leq n$. We will express a choice function f on P_n as $(f(1), \ldots, f(n))$, and a similar expression applies to describe an f-assignment. In fact, since we are only considering choice functions on P_n where $2 \leq f(i) \leq 3$, we may omit the commas and write, for instance, (232232) in place of $(2, 3, 2, 2, 3, 2)$. The *interval* $[i, j]$ refers to the subgraph induced by the vertices $i, i+1, \ldots, j$.

4.1 The color-forcing number

Let f be a choice function on a graph G, and let H be an induced subgraph of G. We will say color a is *forced* on H by an f-assignment $\mathcal{C}$ if for every proper $\mathcal{C}$-coloring c, there exists some vertex $v \in V(H)$ such that $c(v) = a$. In other words, every proper $\mathcal{C}$-coloring must use color a on some vertex of H, although the vertices on which it is used may differ depending on the coloring. For any f-assignment $\mathcal{C}$, we define

$$\mu(H, \mathcal{C}) = |\{a : a \text{ is forced by } \mathcal{C} \text{ on } H\}|,$$
$$\mu(H, f) = \max\{\mu(H, \mathcal{C}) : \mathcal{C} \text{ is an } f\text{-assignment}\}.$$

The *color-forcing number* of G is the maximum of $\mu(G, f)$ over all choice functions f on G. As an example, consider (P_4, g), where $g \equiv 2$. The g-assignment $\mathcal{C} = (12, 12, 34, 56)$ satisfies $\mu(G, \mathcal{C}) = 2$, as only colors 1 and 2 are forced. On the other hand, $\mu(P_4, g) = 4$, as the g-assignment $\mathcal{D} = (12, 12, 34, 34)$ satisfies $\mu(G, \mathcal{D}) = 4$, and this is the maximum, as for any pair (G, f), clearly $\mu(G, f) \leq |V(G)|$ (see Figure 4.2a). Next, consider (C_4, g), where $g \equiv 2$. Let the vertices be v_1, v_2, v_3, v_4, with v_i adjacent to v_{i+1} with addition modulo 4. Let H be the subgraph consisting only of the vertex v_1. The g-assignment $\mathcal{C} = (12, 12, 13, 23)$ satisfies $\mu(H, \mathcal{C}) = 1$ as any proper coloring must use color 1 on v_1, but $\mu(H, \mathcal{C}_H) = 0$ (see Figure 4.2b).

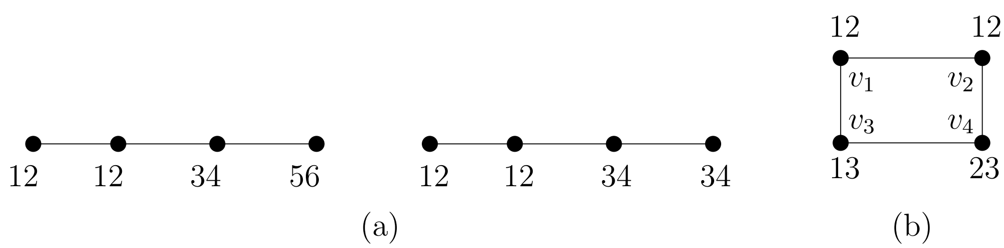


Figure 4.2: The graphs used in the example on forcing colors.

Lemma 4.1. *Let (G, f) be given. Let $v \in V(G)$. Suppose that $(G - v, f_{G-v})$ is choosable. Then (G, f) is choosable if and only if $f(v) > \mu(N(v), f_{G-v})$.*

Proof. Suppose first that (G, f) is choosable. Let $\mathcal{C}'$ be an f_{G-v}-assignment achieving the maximum $\mu(N(v), f_{G-v})$. If, in fact, $f(v) \leq \mu(N(v), f_{G-v})$, then consider an f-assignment $\mathcal{C}$ that satisfies $\mathcal{C}(w) = \mathcal{C}'(w)$ for $w \neq v$, and $\mathcal{C}(v)$ is some subset of the $\mu(N(v), f_{G-v})$ colors that are forced on $N(v)$. We have arrived at a contradiction as $\mathcal{C}$ can have no proper coloring.

Conversely, suppose that $f(v) > \mu(N(v), f_{G-v})$. Let $\mathcal{C}$ be an f-assignment. Since $(G - v, f_{G-v})$ is choosable, there exists a proper $\mathcal{C}_{G-v}$-coloring of $G - v$. Further, for any proper $\mathcal{C}_{G-v}$-coloring c of $G - v$, $\mathcal{C}(v) \backslash \{c(w) : w \in N(v)\}$ is nonempty, since otherwise, $\mu(G - v, \mathcal{C}) \geq f(v)$. Thus c can be extended to a proper $\mathcal{C}$-coloring of F_n. $\square$

By the above lemma, (F_n, f) is choosable if and only if $f(0) > \mu(P_n, f_{[1,n]})$. Thus we will focus on computing $\mu(P_n, g)$ for size functions g on P_n. Recall that here we are interested in those size functions g that satisfy $g(1) = g(n) = 2$ and $2 \leq g(i) \leq 3$ for all $1 < i < n-1$. Such size functions will be called *basic choice functions*. It suffices to restrict our attention to basic choice functions, as we are looking for choice functions with size less than the greedy bound, but we are not determining the sum choice number.

4.2 Forcing colors on paths

The following lemma gives a choosability characterization for the path P_n, which is probably well-known, though perhaps not documented.

Lemma 4.2. *Let f be a size function on P_n with $f(i) > 0$ for each $i = 1, \ldots, n$. Then f is choosable if and only if there do not exist indices $i_1 < i_2$ with $f(i_1) = f(i_2) = 1$ and $f(i) = 2$ for $i_1 < i < i_2$. Moreover, in the case that f is not choosable, any f-assignment which has no proper coloring must have a restriction to $[i_1, i_2]$ of the form $(a_1, a_1a_2, a_2a_3, \ldots, a_{q-1}a_q, a_q)$, where $q = i_2 - i_1$.*

Proof. If there exist vertices $i_1 < i_2$ with $f(i_1) = f(i_2) = 1$ and $f(i) = 2$ for $i_1 < i < i_2$, we will say that f has the *bad subgraph property with end vertices i_1 and i_2*. If f has the bad subgraph property, then there is clearly no proper coloring from any f-assignment $\mathcal{C}$ satisfying $\mathcal{C}(i_1) = 1$, $\mathcal{C}(i) = 12$ for $i_1 < i < i_2$, and $\mathcal{C}(i_2) = 2$, if $i_2 - i_1$ is even, and $\mathcal{C}(i_2) = 1$, if $i_2 - i_1$ is odd. We now prove the converse, that if f does not have the bad subgraph property, then it is choosable. We do this by induction on n, with trivial basis $n = 2$ (the lemma is clearly true for $n = 1$). Assume the result for P_{n-1}. Let f be a size function on P_n that does not satisfy the bad subgraph property, let $\mathcal{C}$ be an f-assignment, and let S denote the subgraph induced by the vertices $1, \ldots, n-1$. If $f(n) > 1$, then f is choosable, since by the induction hypothesis, there exists a proper $\mathcal{C}$-coloring c of S, and this extends to a proper coloring of P_n since $\mathcal{C}(n) \backslash \{c(n-1)\} \neq \emptyset$. On the other hand, if $f(n) = 1$, then by Lemma 2.1, f is choosable if and only if f^n is choosable. We shall assume f^n is not choosable, and arrive at a contradiction. If f^n is not choosable, then by the induction hypothesis, it satisfies the bad subgraph property for some end vertices j_1 and j_2. As f does not have the bad subgraph property, it must be that $j_2 = n - 1$, but then $f(n-1) = 2$, and hence f does indeed satisfy the bad subgraph property with end vertices j_1 and n. This is a contradiction with our assumption on f, so f^n must be choosable, and hence, f is also choosable.

We now prove the second statement of the lemma by induction on n. The basis $n = 2$ is clear. Assume the result holds on P_{n-1}. Let $\mathcal{C}$ be an $f_{[i_1,i_2]}$-assignment having no proper coloring, and let a be the lone color of $\mathcal{C}(i_2)$. By the first part of the lemma, the restriction of $\mathcal{C}$ to the vertices $i_1, \ldots, i_2 - 1$ has a proper coloring. If $a \notin \mathcal{C}(i_2 - 1)$, then this can be extended to a proper coloring of all of $[i_1, i_2]$. So it must be that $a \in \mathcal{C}(i_2 - 1)$. In this case, there does not exist a proper $\mathcal{C}$-coloring of $[i_1, i_2]$ if and only if there does not exist a proper coloring of $[i_1, i_2]$ from the list assignment $\mathcal{C}'$ obtained from $\mathcal{C}$ by removing color a from $\mathcal{C}(i_2 - 1)$. Further, by the induction hypothesis, this is true if and only if $\mathcal{C}'$ is the form given in the statement of the lemma. However, it then follows that $\mathcal{C}$ is of that form also. $\square$

We now use the above choosability characterization to give a color-forcing characterization. Figure 4.3 gives an example of the lists in the statement of the following lemma. For clarity, we display the vertices with list size 2 and 3 at different levels.

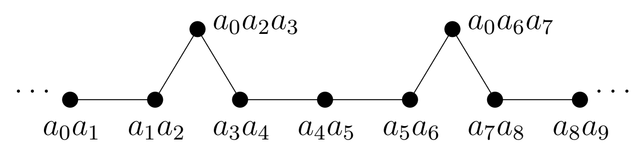


Figure 4.3: An example of the lists for Lemma 4.3.

Lemma 4.3. *Let f be a basic choice function on P_n, and let $\mathcal{C}$ be an f-assignment. Color a_0 is forced by $\mathcal{C}$ on P_n if and only if there exist indices $k_0 < k_1$ such that for $j = 0, \ldots, q-1$, the f-assignment $\mathcal{C}$ satisfies $\mathcal{C}(k_0 + j) = a_j a_{j+1}$ if $f(k_0 + j) = 2$, $\mathcal{C}(k_0 + j) = a_0 a_j a_{j+1}$ if $f(k_0 + j) = 3$, and $\mathcal{C}(k_1) = a_0 a_q$, for some colors $a_1, \ldots, a_q$, where $q = k_1 - k_0$.*

Proof. Assume first that the condition on $\mathcal{C}$ holds. Then any proper $\mathcal{C}$-coloring c must satisfy $c(k_0) = a_0$ or $c(k_1) = a_0$, since if $c(k_0) \neq a_0$, then by tracing through the lists, one easily concludes that $c(k_1) = a_0$. Conversely, suppose $\mathcal{C}$ does not satisfy the condition. The only way in which all proper $\mathcal{C}$-colorings would use a_0 on some vertex is if the lists with a_0 removed are in the form prescribed in Lemma 4.2. This happens only when the $\mathcal{C}$ satisfies the condition given in this lemma. Therefore, there exists a proper $\mathcal{C}$-coloring avoiding a_0, and hence a_0 is not forced. $\square$

Let k_0 and k_1 be the minimum vertices for which the lists appear as they do in the above lemma. We will call k_0 the *initial vertex* of a_0 and k_1 the *final vertex*. We refer to lists in the form given in the lemma as a *string* (for a_0). Thus, we may say that k_0 is the minimum vertex for which the lists starting at k_0 are a string for a_0, and k_1 is the minimum vertex greater than k_0 with list size 2 that has a_0 on its list. Given a choice function f on P_n, set $t(P_n, f) = |\{i : f(i) = 2\}|$.

Lemma 4.4. *If f is a basic choice function on P_n, then $\mu(P_n, f) \leq t(P_n, f)$.*

Proof. By Lemma 4.3, the initial and final vertices of any color a that is forced must be distinct and must be assigned list size 2 by f. $\square$

We imagine that there are $2t(P_n, f)$ *slots* to be filled with colors, two slots for each vertex assigned list size 2. For an f-assignment to force $t(P_n, f)$ colors, each slot must be utilized as either the initial or the final vertex of a (forced) color. That is, each vertex of list size 2 must be either the initial vertex for two colors, the final vertex for two colors, or the the initial vertex for one color, and the final vertex for another color. When we say a slot on vertex i is *unused*, we mean that vertex i is initial for at most one color, final for at most one color, and not initial for one color and final for another. When we say both slots on vertex i are unused, we mean that vertex i is not initial or final for any color. We now prove three elementary lemmas that will be quite useful.

Lemma 4.5. *If a vertex i is initial for two colors, and $f(i+1) = 2$, then $i+1$ is final for those two colors.*

Proof. This follows directly from Lemma 4.3. □

Lemma 4.6. *Let f be a basic choice function on P_n, and let $\mathcal{C}$ be an f-assignment. If $\mathcal{C}(j) = ab$ and $\mathcal{C}(j+1) = bc$ for some vertex j and some colors a, b, and c with $a \neq c$, then a slot is unused on either j or $j+1$.*

Proof. By Lemma 4.3, it cannot happen that j is the initial vertex of b and $j+1$ is the final vertex of b. □

Lemma 4.7. *Let f be a basic choice function on P_n, let $\mathcal{C}$ be an f-assignment, and let a be a color. Set $s = |\{i : a \in \mathcal{C}(i)\}|$. If $s > 2$, then there are at least $s-2$ slots unused. If $s = 1$, then there is a slot unused.*

Proof. Only one vertex can be initial for a, and only one vertex can be final for a. Moreover, a must appear on the lists of at least two vertices in order to be forced. □

Lemma 4.8. *Let $f \equiv 2$ be a basic choice function on P_n. Then $\mu(P_n, f) = 2\lfloor n/2 \rfloor$.*

Proof. First, the maximum is achieved by the f-assignment $\mathcal{C}$ satisfying $\mathcal{C}(v_i) = \mathcal{C}(v_{i+1}) = a_i a_{i+1}$ for $i = 1, 3, 5, \ldots$, where the a_i are any distinct colors. On the other hand, if $\mu(P_n, f) = n$, then each slot must be utilized. This means that vertex 1 must be initial for two colors, and hence, by Lemma 4.5, vertex 2 must be the final vertex of those two colors. Repeating this argument, we see that the vertices $3, 5, \ldots, 2n-1$ each must be initial for two colors, which is not possible if n is odd. Thus $\mu(P_n, f) < n$ if n is odd, and in fact, the argument shows that $\mathcal{C}$ is unique if n is even. □

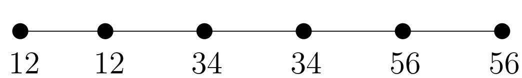


Figure 4.4: The list assignment $\mathcal{C}$ for the even case in Lemma 4.8

Lemma 4.9. *Let $f \equiv 2$ be a choice function on P_n. Let $\mathcal{C}$ be an f-assignment of the form $(a_0a_1, a_1a_2, a_2a_3, \ldots a_{n-1}a_n)$. Then $\mu(P_n, \mathcal{C}) = \lfloor 2(n+1)/3 \rfloor$.*

Proof. First, we give lists achieving the maximum. We will use integers for the colors. The lists for the cases $n = 2$, 3, and 4 are $(12, 12)$, $(12, 12, 13)$, and $(12, 12, 13, 13)$, respectively. Set $k = \lfloor 2(n+1)/3 \rfloor$. If $n \equiv 2 \pmod 3$, we get the lists for case n from the lists for case $n-2$ by appending the lists $\{k-1, k\}$, $\{k-1, k\}$. If $n \equiv 0 \pmod 3$, we get the lists for case n from the lists for case $n-1$ by appending the list $\{k-1, k+1\}$. If $n \equiv 1 \pmod 3$, we get the lists for case n from the lists for case $n-1$ by appending the list $\{k-2, k\}$. For example, we have

$$\begin{aligned} n = 5 &: (12, 12, 13, 34, 34), \\ n = 6 &: (12, 12, 13, 34, 34, 35), \\ n = 7 &: (12, 12, 13, 34, 34, 35, 35). \end{aligned}$$

We now show that $\lfloor 2(n+1)/3 \rfloor$ is the upper bound. Let a be a color whose initial and final vertices differ by at least two. By Lemma 4.3, if a has initial vertex greater than 1 and final vertex less than n, then at least four slots must be used in the forcing of a. That is to say, a must appear on the lists of at least four different vertices. If the initial vertex is 1 or the final vertex is n, then at least three slots must be used. Let b be a color with initial vertex i and final vertex $i+1$. By Lemma 4.5, if the list on v_i is bc, then the list on v_{i+1} must also be bc. Assume first that $1 < i < n-1$. Then by Lemma 4.3, one of the colors b and c must appear on the list for v_{i+1} and by the hypothesis on the lists in the current lemma, that color must also appear on the list for v_{i-1}. Thus, together the colors b and c use up at least six slots — two for one color, four for the other. If $i = 1$ or $i = n-1$, then a similar argument shows that the colors b and c together use up at least five slots — two for one color, three for the other. We conclude that the most efficient use of slots is, as much as possible, to have a color's initial and final vertices be adjacent. Moreover, it is most efficient for a list assignment to force two colors with initial vertex 1 and two colors with initial vertex $n-1$. Thus, assuming we force two colors with initial vertex 1 and two colors with initial vertex $n-1$, we know from the hypothesis and Lemma 4.3 that no color can have 3 or $n-2$ as either its initial or final vertex. Each further color that is forced must have initial vertex i satisfying $3 < i < n-2$. Thus, there remain $2n-10$ slots unfilled, and each pair of colors forced requires a total of at least six slots, according the the above discussion. We can thus force at most $\lfloor (2n-10)/6 \rfloor$ pairs of colors, in addition to the four already forced, bringing the total number of colors forceable up to $4 + 2\lfloor (n-5)/3 \rfloor$. There are 0, 2, or 4 slots left over after forcing pairs for $n \equiv 2, 1, 0 \pmod 3$, respectively. Thus in the case $n \equiv 1 \pmod 3$, there is room for one more color to be forced, but not in the other two cases, by the above discussion about forcing a single color. Thus, for n congruent to 0 or 2 modulo 3, there are at most $4 + 2\lfloor (n-5)/3 \rfloor$ colors that can be forced, and at most $5 + 2\lfloor (n-5)/3 \rfloor$ colors if n is congruent to 1. In all cases, it can be checked that these quantities simplify to $\lfloor 2(n+1)/3 \rfloor$. ∎

We make the following definitions. Let f_i be a choice function on P_{i+4} such that $f_i(2) = f_i(i+3) = 3$ and $f_i(j) = 2$ for all other vertices j. For example, $f_3 = (2322232)$ and $f_4 = (23222232)$. For $i = 1, \ldots, 5$ we define the following f_i assignments, L_i.

$L_1[a_1, a_2, a_3] = (a_1a_2, a_1a_2a_3, a_1a_3, a_1a_2a_3, a_2a_3)$,
$L_2[a_1, a_2, a_3, a_4] = (a_3a_4, a_1a_3a_4, a_1a_2, a_1a_2, a_1a_3a_4, a_3a_4)$,
$L_3[a_1, a_2, a_3, a_4] = (a_3a_4, a_1a_3a_4, a_1a_2, a_1a_2, a_1a_2, a_2a_3a_4, a_3a_4)$,
$L_4[a_1, a_2, a_3, a_4, a_5] = (a_4a_5, a_1a_4a_5, a_1a_2, a_1a_2, a_1a_3, a_1a_3, a_1a_4a_5, a_4a_5)$,
$L_5[a_1, a_2, a_3, a_4, a_5, a_6] = (a_5a_6, a_1a_5a_6, a_1a_2, a_1a_2, a_1a_3, a_3a_4, a_3a_4, a_3a_5a_6, a_5a_6)$,

where $a_1, \ldots, a_6$ are distinct colors (see Figure 4.5). One can easily check that each of the above lists forces every color in their argument, e.g., $L_1[a_1, a_2, a_3]$ forces a_1, a_2, and a_3. Given (P_n, f) with $f \equiv 2$ and n even, we define the f-assignment $\mathcal{P} =$

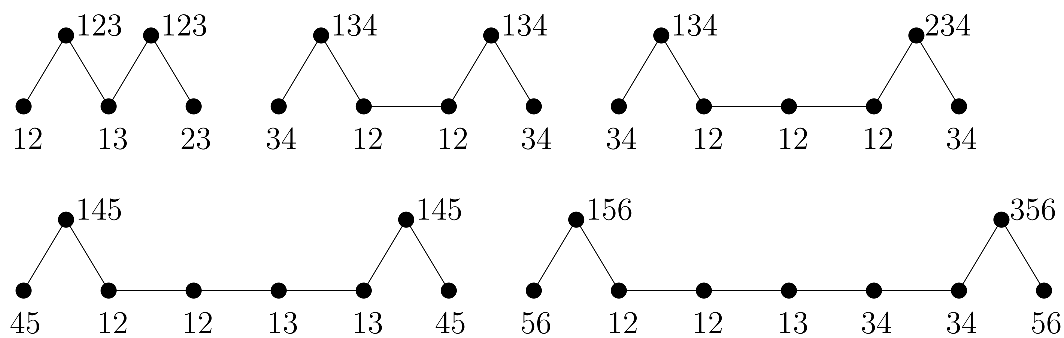


Figure 4.5: The list assignments L_1, L_2, L_3, L_4, and L_5.

$P[a_1, \ldots, a_n]$ by $\mathcal{P}(v_i) = \mathcal{P}(v_{i+1}) = a_ia_{i+1}$ for $i = 1, 3, 5, \ldots, n-1$, where the a_i are any distinct colors. We think of this as a "pairing off" of the vertices (see, for example, Figure 4.4). Given (P_3, f) with $f = (232)$, we define the f-assignment $S[a_1, a_2, a_3] = (a_1a_2, a_1a_2a_3, a_2a_3)$. One can easily check that $P[a_1, a_2, \ldots, a_n]$ forces every one of those colors, and $S[a_1, a_2, a_3]$ forces only color a_2.

Lemma 4.10. *If $i \leq 2$, then $\mu(P_{i+4}, f_i) = t(P_{i+4}, f_i)$. If $3 \leq i \leq 5$, then $\mu(P_{i+4}, f_i) = t(P_{i+4}, f_i) - 1$. If $i > 5$, then $\mu(P_{i+4}, f_i) = t(P_{i+4}, f_i) - 2$.*

Proof. One may easily verify that for $i \leq 5$, the lists L_i achieve the bound. Let $a_1, a_2, \ldots, a_{i+2}$ be distinct colors. If $i > 5$ and even, then any f_i-assignment that is equal to $P[a_1, a_2, \ldots, a_i]$ on $[3, i+2]$ achieves the bound. If $i > 5$ and odd, then any f_i assignment that is equal to $P[a_1, a_2, \ldots, a_{i-1}]$ on $[3, i+2]$ and equal to $S[a_i, a_{i+1}, a_{i+2}]$ on $[1, 3]$ achieves the bound. It remains to show that this bound is best possible. It is clearly best possible for $i = 1, 2$ by Lemma 4.4. Let $i > 2$ and let $\mathcal{C}$ be an f_i-assignment.

Suppose first that color a_1 is forced by $\mathcal{C}$ with initial vertex 1 and final vertex j in $[3, i+2]$. We will show that some slot is unused. First, suppose $j = 3$. By Lemma

4.3, 3 is not final for another color. If 3 is initial for some color, then $a_1 \in \mathcal{C}(4)$, so by Lemma 4.7 a slot is unused. Suppose then that $j > 3$. Note that $a_1 \notin \mathcal{C}(j-1)$, since otherwise $j-1$ would be final for a_1. Whatever the other color on $\mathcal{C}(j)$ is, it must also be on $\mathcal{C}(j-1)$, so by Lemma 4.6, a slot is unused on either $j-1$ or j.

Suppose next that color a_1 is forced by $\mathcal{C}$ with initial vertex j in $[3, i+2]$ and final vertex $i+4$. We will show that some slot is unused. Let a_2 be the other color on $\mathcal{C}(i+2)$. First, suppose $j = i+2$. Since $i+4$ is final for a_1, $\mathcal{C}(i+1) \neq a_1a_2$ by Lemma 4.5. By Lemma 4.3, $i+2$ is not initial for a_2. If $i+2$ were final for a_2, then $a_1 \in \mathcal{C}(i+1)$, and a slot is unused on $i+1$. Otherwise, a slot is unused on $i+2$. If $j < i+2$, then $a_2 \in \mathcal{C}(j+1)$ and by Lemma 4.5 $a_1 \notin \mathcal{C}(j+1)$, since $i+4$ is final for a_1. Thus, by Lemma 4.6, a slot is unused on either j or $j+1$.

Suppose now that $\mathcal{C}_{[3,i+2]}$ is not a string. Let i_0 be the maximum vertex for which $\mathcal{C}_{[3,i_0]}$ is a string. That is, i_0 is such that there is no color forced with initial vertex at most i_0 and final vertex greater than i_0. Note that $3 \leq i_0 < i+2$ since $\mathcal{C}_{[3,i+2]}$ is not a string. By Lemma 4.3, and since $\mathcal{C}_{[3,i+2]}$ is not a string, any color forced with final vertex $i+4$ must have an initial vertex greater than i_0. If there are no colors with final vertex $i+4$, then two slots on $i+4$ are unused, and if there is exactly one color with final vertex $i+4$, then there is one slot on $i+4$ unused. If there is a color with final vertex $i+4$, then by the argument of the third paragraph, there is a slot unused in $[i_0+1, i+2]$. Similarly, if there are no colors forced with vertex index 1, then two slots on 1 are unused, and if there is exactly one color with initial vertex 1, then there is one slot on 1 unused. If there is a color with initial vertex 1, then by the argument of the second paragraph, there is a slot unused in $[3, i_0]$. Combining these, we see that there are in total at least three slots unused, except possibly if there are two colors with initial vertex 1 and two colors with final vertex $i+4$. As mentioned, no color has initial vertex 1 and final vertex $i+4$, so the final vertices of the colors on $\mathcal{C}(1)$ must be in $[3, i_0]$. Let k_1 and k_2, $k_1 < k_2$ be the final vertices of the colors on $\mathcal{C}(1)$. Note that $k_1 \neq k_2$ by Lemma 4.3. By Lemma 4.6, a slot is unused on k_1+1, and if $k_2 \neq i_0$, then by Lemma 4.6, a slot is unused on k_2+1. If $k_2 = i_0$, then by Lemma 4.5, k_2 is not final for the other color on $\mathcal{C}(k_2)$ and is not initial for the color by the definition of i_0. Therefore, a slot is unused on k_2. In all cases, there are at least three slots unused in total.

Suppose then that $\mathcal{C}_{[3,i+2]}$ is a string. By checking a small number of possibilities, one can conclude from Lemmas 4.6 and 4.7 that at least two slots are unused in $[3, 5]$, and further, that at least three slots are unused in $[3, 7]$ unless $\mathcal{C}_{[3,7]} = (a_3a_4, a_3a_4, a_3a_5, a_5a_6, a_5a_6)$ and $i = 5$. If $i > 5$ and $\mathcal{C}$ restricts to the above lists on $[3, 7]$, then a third slot is lost on 8. The result follows. $\square$

Note that $\mathrm{GB}(F_n) = 3n$. We now get to the main result.

Theorem 4.11. *Fan graphs are not sc-greedy in general, and in fact, the gap between* $\mathrm{GB}(F_n)$ *and* $\chi_{\mathrm{SC}}(F_n)$ *can be made arbitrarily large.*

Proof. By the preceding lemma $\mu(P_{10}, f_6) = 6$. Thus by Lemma 4.1, the size function g on F_{10} defined by $g(v_0) = 7$ and $g(v_i) = f_6(v_i)$ for $i > 0$ is choosable. Note that $\text{size}(g) = 29$, but $\text{GB}(F_{10}) = 30$. Moreover, one can show that the gap between $\text{GB}(F_n)$ and $\chi_{\text{SC}}(F_n)$ can be arbitrarily large. For example, consider the size function h_k on P_{11k-1} defined by $h_k(v_i) = 3$ if $i \equiv 2$ or 9 modulo 11, and $h(v_i) = 2$ for any other i (see Figure 4.6 and recall that we display the vertices with list size 3 on a level above those of list size 2). Then $\mu(P_{11k-1}, h_k) = t(P_{11k-1}, h_k) - k - 1$, and hence the size function g' on F_{11k-1} defined by $g'(v_0) = t(P_{11k-1}, h_k) - k$ and $g'(v_i) = h_k(v_i)$ for $i > 0$ is choosable and of size $\text{GB}(F_{11k-1}) - k$. We give a sketch of why $\mu(P_{11k-1}, h_k) = t(P_{11k-1}, h_k) - k - 1$. First, note that by the arguments of the second and third paragraphs of the preceding lemma, at least two slots are lost at each of the ends; that is, at least two slots are unused in $[1, 8]$, and at least two slots are unused in $[11k - 8, 11k - 1]$. Consider now the intervals $[11j - 1, 11j + 1]$ for $1 \leq j < k$. By arguments similar to those used in the preceding lemma, we can show that at least two slots are lost within each of these intervals, unless the preceding interval $[11j - 8, 11j - 3]$ is a string. In that case, however, at least two slots are lost on that interval, in addition to any that were already lost there. In total, there are at least $2(k + 1)$ slots unused, and hence at most $t(P_{11k-1}, h_k) - (k + 1)$ colors can be forced. $\square$

In particular, F_{10} is not sc-greedy, thus answering the question of Isaak, Pelsmajer, and Albertson: Outerplanar graphs are in general not sc-greedy. We suspect that the value $\text{GB}(F_n) - \lfloor (n + 1)/11 \rfloor$ is actually the sum choice number. That is, the gap between the greedy bound and the sum choice number jumps up by one when n increases from $n = 11k - 2$ to $n = 11k - 1$ for $k > 1$, and the gap stays the same otherwise. So the pairs (P_{11k-1}, h_k) may, in some sense, be minimal. One can verify quite easily using the ideas given herein that F_n is sc-greedy for $n < 10$. In fact, the ideas of the proof of Lemma 4.10 can likely be used to give a polynomial algorithm to determine if a given size function on F_n is choosable.

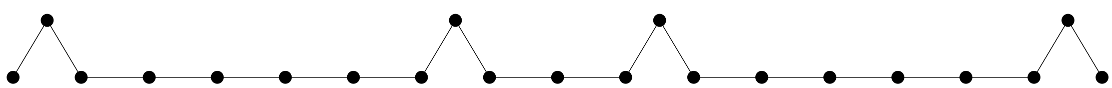


Figure 4.6: The choice function h_2 on P_{21}

Chapter 5

Conclusions and Directions for Future Work

The determination of the sum choice number of $P_3 \square P_n$ in Chapter 3 is of a rather unfortunate length. The proof of the upper bound is reasonably short, and could possibly be improved a bit, but the only method of proof for the lower bound that we found was to proceed in (numerous) cases. We used the same essential idea to eliminate many of the cases, but there still remained several special cases that had to be dealt with on an individual basis. There may be some way to quickly eliminate from consideration any size function that assigns size 1 to some vertex, which would then considerably cut down on the work involved. There seems to be some hope for the $n \times m$ grid $P_n \square P_m$. We have made some progress in determining $\chi_{\mathrm{SC}}(P_4 \square P_n)$; any hints at a formula for the general case, however, have eluded us.

The techniques of Chapter 4 can be used to determine the sum choice number of fan graphs, and further can also give a fast algorithm to determine if a given size function on a fan graph is choosable. All of this relies on the computation of the color-forcing number of P_n. However, though we believe a relatively straightforward computation exists, all attempts thus far have resulted in ugly arguments with numerous cases. The determination of the color-forcing number for graphs other than paths may be interesting, but likely difficult.

A natural generalization of the Peterson graph result is to consider the sum choice number of Kneser graphs. The Kneser graph $K(n, k)$ is the graph whose vertices correspond to the k-element subsets of $\{1, 2, \ldots, n\}$, with an edge between two vertices if and only if their corresponding subsets are disjoint. The Peterson graph is $K(5, 2)$, and its abundant symmetry of the Peterson graph helped to keep the proof relatively short. This may carry over to more general Kneser graphs, in particular those of the form $K(p, 2)$.

Though not included here, we have done a considerable amount of work on generalized theta graphs, graphs consisting of two vertices joined by k internally vertex disjoint paths. We expect some results on sum choice numbers and possibly a choosability characterization. Note that $K_{2,n}$ is a generalized theta graph, and in fact, ideas similar to the blocking idea of Lemma 2.9 are quite useful here. We have determined that choosability of generalized theta graphs depends solely on the number of paths with one internal vertex, the number of paths with an even number of internal vertices, and the number of paths with an odd number (greater than one) of internal vertices.

The author and Arthur Busch have investigated the sum choice number of the general complete bipartite graph $K_{p,q}$, for which Lemma 2.9 is still quite useful. However, the important fact from the $p = 3$ case, that for any minimum choice function, the lists on the larger partite set all have size 2, seems to break down for $p > 4$. It still seems that this is the most important case of the proof, and in fact, we may have a characterization of choosability, for any p and q, if the lists on the larger partite set are all 2. One should not expect the determination of the sum choice number of $K_{p,q}$ to be easy, given that determination of the ordinary choice number on $K_{p,q}$ is so difficult. On the other hand, the choice number of complete multipartite graphs with each partite set of size 2 has been determined [3], as has the case when all partite sets are of size 3 [4]. Thus, it seems reasonable to compute the sum choice number for these graphs.

Lemma 2.9 may have significant applications. One application is to determine $\chi_{\mathrm{SC}}(G \vee \overline{K}_q)$, where G has a particularly simple structure. Recall that $K_{p,q} = \overline{K}_p \vee \overline{K}_q$. We have some results for arbitrarily large q in the case where G is K_2, K_3, or P_3. It seems that further work could produce some interesting results. However, we should not expect too much if G is very large. For example, the fan graph F_n is $P_n \vee \overline{K}_1$, and this has already proven to be nontrivial. On the other hand, we may be able to determine $\chi_{\mathrm{SC}}(K_p \vee \overline{K}_q)$ without too much difficulty.

As mentioned in the discussion following Theorem 2.5, the techniques of that proof extend to show that graphs created by laying end-to-end cycles of arbitrary and varying lengths greater than 3 are sc-greedy. As mentioned there, if we laid the cycles along a tree structure or a cycle, instead of end-to-end, the resulting graph would still be sc-greedy. We believe that if cycles of length 3 are allowed, then the resulting graphs would still be sc-greedy, but a more delicate argument would be required (compare to Theorem 2.6). It may even be true that if cycles were laid out rather than end-to-end or on a tree structure, but on the structure of an sc-greedy graph, the resulting graph would still be sc-greedy (see Figure 5.1).

It seems that vertices of high degree have the potential to wreak havoc with the sum choice number. This is evidenced by fan graphs and complete bipartite graphs. In Chapter 4, we noted that choosability of paths is a simple matter, but as we saw, as soon as one joins a vertex to the path (thus creating a vertex of high degree), things become more difficult. The extreme simplicity of the structure of $K_{2,q}$ prevented this from becoming an issue, but the influence becomes more apparent with $K_{3,q}$. It would be interesting to see what effect vertices of high degree have on other graphs.

It appears that much of the time, the determination of the sum choice number of a graph is more difficult than the determination of the list chromatic number. To

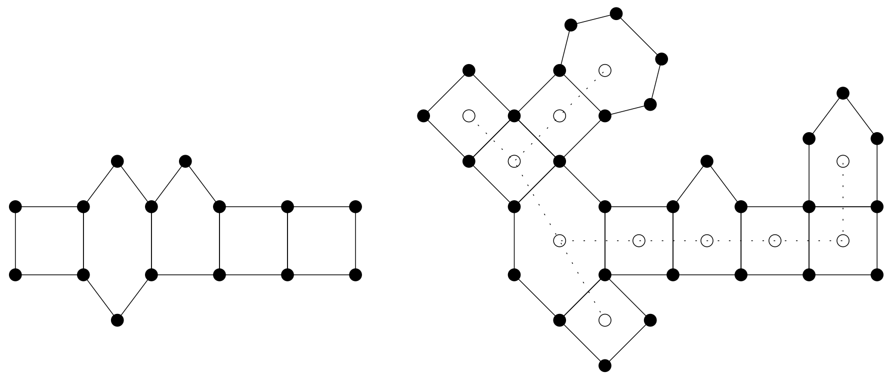


Figure 5.1: Cycles laid end-to-end or along an underlying tree structure

show that the list chromatic number of a graph is k, one has to show that $f \equiv k$ is choosable, whereas $f \equiv k - 1$ is not. In all, only two size functions need to be considered. However, to get a lower bound for the sum choice number, one may have to consider quite a few size functions. An illustrative example is $P_3 \square P_2$, which we have considered on a number of occasions. On the other hand, as mentioned in the introduction, the determination of the list chromatic number for complete bipartite graphs is quite hard, but the determination of the sum choice number may turn out to be tractable. To see why this is true, note that it is intuitively reasonable that to find a choice function of minimum size, one should assign more list size to the vertices of the smaller partite set (whose vertices have higher degree) than to vertices of the larger partite set. In fact, the minimum choice functions constructed in Theorems 2.10 and 2.12 assign list size 2 to every vertex of the larger partite set, and the list sizes of the vertices of the smaller partite set grow arbitrarily large as the graphs get larger. Thus, in the case of complete bipartite graphs, sum list coloring bypasses some of the difficulties encountered in the determination of the list chromatic number. In particular, determining if there is a proper coloring from lists is less difficult when a large number of the lists have size 2. In fact, if all the lists are of size 2, then the existence of a proper coloring from the lists can be found quickly, by choosing a color on one of the lists, and then tracing through the possibilities.

One ambitious project would be to characterize sc-greedy graphs. Another possibility would be the calculation of the sum choice number for a general outerplanar graph. It would be interesting to find heuristics that do better than the greedy algorithm on reasonably broad classes of graphs. There are algorithms and techniques developed in the study of the list chromatic number, but in the case of minimizing the sum of the list sizes, none of these ever seems to do better than the greedy algorithm. We would also be interested in results concerning the difference $\chi_{SC}(G) - \mathrm{GB}(G)$ for various graphs. We suspect that this difference is maximized by complete bipartite graphs. One can check, for example, that $K_{2,3}$ is the smallest (in terms of number of vertices) graph which is not sc-greedy.

In conclusion, we feel that previous results and our results have only scratched the surface of what may be a very interesting field of research. The techniques developed herein can be used to determine sum choice numbers of other classes of graphs, and have applications to the broader question of choosability.

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Vita

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