Patterns and Fractals from a Generalized Bitwise AND

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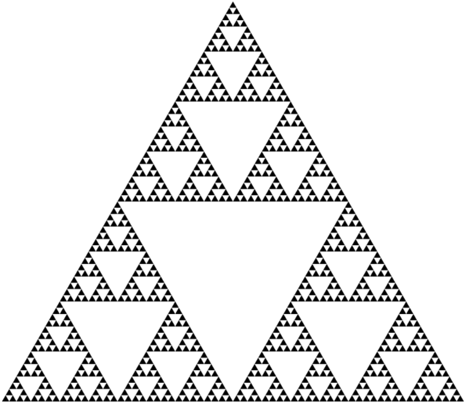
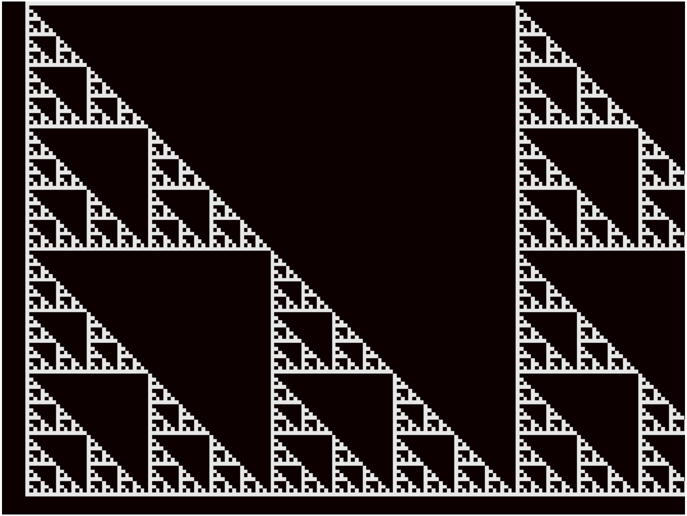


A Qbasic Program

```
SCREEN 12
FOR X = 1 TO 640
    FOR Y = 1 TO 480
        PSET (X, X AND Y), 15
    NEXT Y
NEXT X
```

The Output (more or less)



Bitwise AND

$1 \text{ AND } 1 = 1$
$1 \text{ AND } 0 = 0$
$0 \text{ AND } 1 = 0$
$0 \text{ AND } 0 = 0$

Example: 12 AND 7

```
12 -> 1100
 7 -> 0111
----------
 4 <- 0100
```

The previous slide is a plot of $\{(x, y) : x \text{ AND } y = 0\}$.

A Generalization

Previous slide: 12 AND 7 (base 2)

```
12 -> 1100
 7 -> 0111
----------
 4 <- 0100
```

A Generalization

Previous slide: 12 AND 7 (base 2)

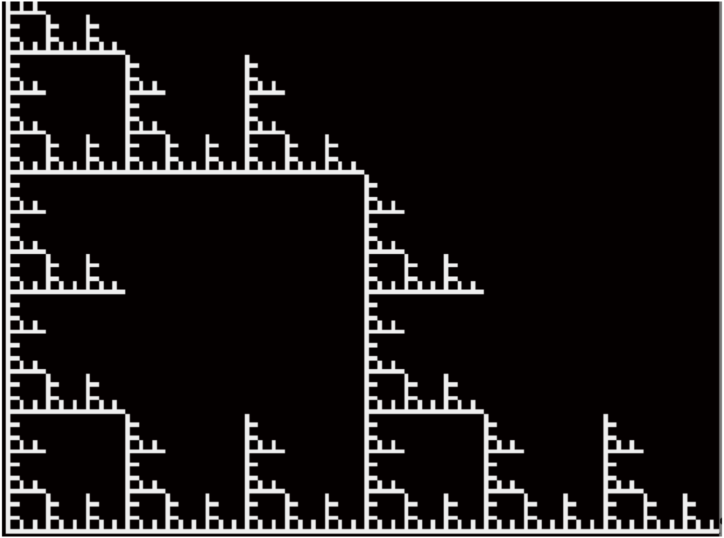
```
12 -> 1100
 7 -> 0111
----------
 4 <- 0100
```

New example: 12 AND_3 7 (base 3)

```
12 -> 110
 7 -> 021
----------
 6 <- 020
```

Convert to base 3, multiply digit-wise without carrying, and convert back to base 10.

$x \text{ AND}_3 \ y$



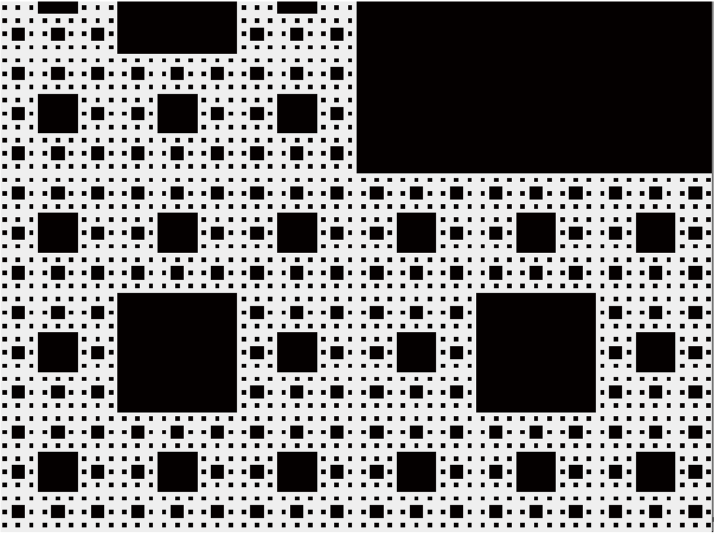
One More Twist

```
12 -> 110
 7 -> 021
----------
      020

mod each digit by 2

 0 <- 000
```

A Surprise? x AND$_3$ y



The General Operation

For $x, y \geq 0$, the general operation is denoted $x \text{ AND}_j^k\ y$.

It is computed by first writing x and y in base j as
$x = (x_n x_{n-1} \ldots x_1)_j$ and $y = (y_n y_{n-1} \ldots y_1)_j$

(where $n = \lfloor \log_j(\max\{x, y\}) \rfloor$ and some of the leading digits of x or y may be 0).

Then compute $d_i = x_i \times y_i \mod k$ for $i = 1, \ldots, n$.

Finally, construct $d = (d_n d_{n-1} \ldots d_1)_j$ and the result is $(d)_{10}$.

We will be plotting $\{(x, y) : x \text{ AND}_j^k y = 0\}$

Example: 510 AND_7^6 274

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First, convert to base 7:

$510 \rightarrow (1326)_7$
$274 \rightarrow (541)_7$

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$274 \rightarrow (541)_7$

Then, do the multiplication:

```
1  3  2  6
0  5  4  1
----------
0 15  8  6
```

Example: 510 AND_7^6 274

First, convert to base 7:

$510 \rightarrow (1326)_7$
$274 \rightarrow (541)_7$

Then, do the multiplication:

```
1  3  2  6
0  5  4  1
----------
0 15  8  6
```

Next, mod by 6:

$0\ 15\ 8\ 6 \rightarrow 0320$

Example: 510 AND_7^6 274

First, convert to base 7:

$510 \rightarrow (1326)_7$
$274 \rightarrow (541)_7$

Then, do the multiplication:

```
1  3  2  6
0  5  4  1
----------
0 15  8  6
```

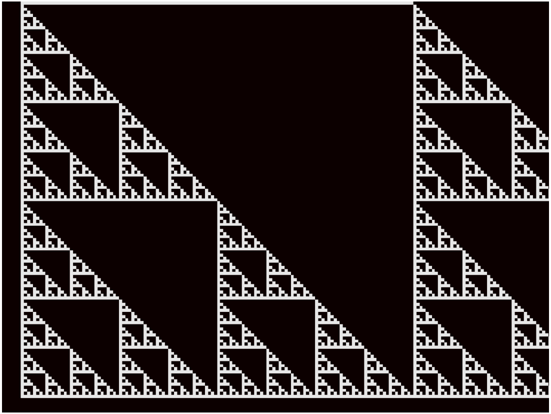
Next, mod by 6:

$0\ 15\ 8\ 6 \rightarrow 0320$

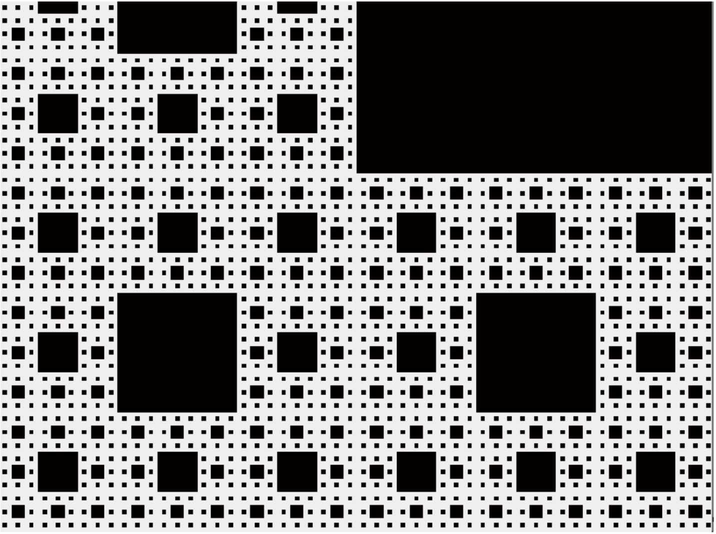
Finally, convert back to base 10:
$(320)_7 \rightarrow 161.$

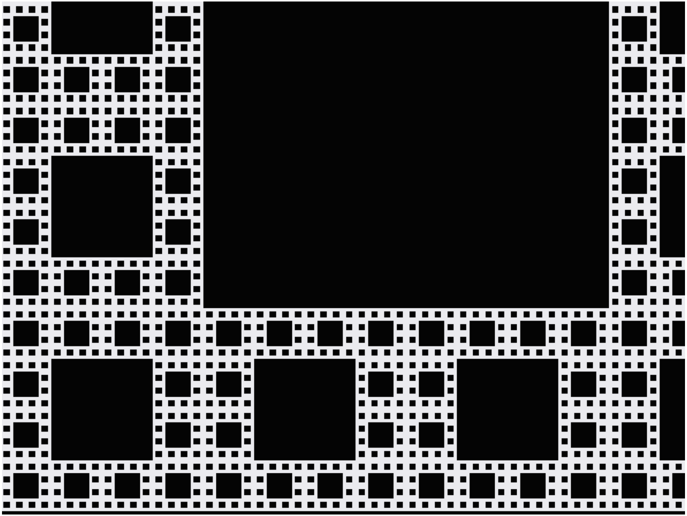
A Look Back at x AND y

0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000

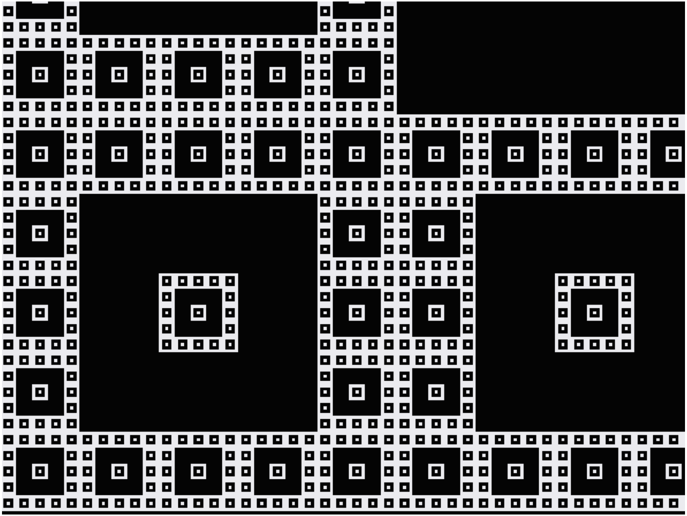


A Look Back at x AND$_3^2$ y

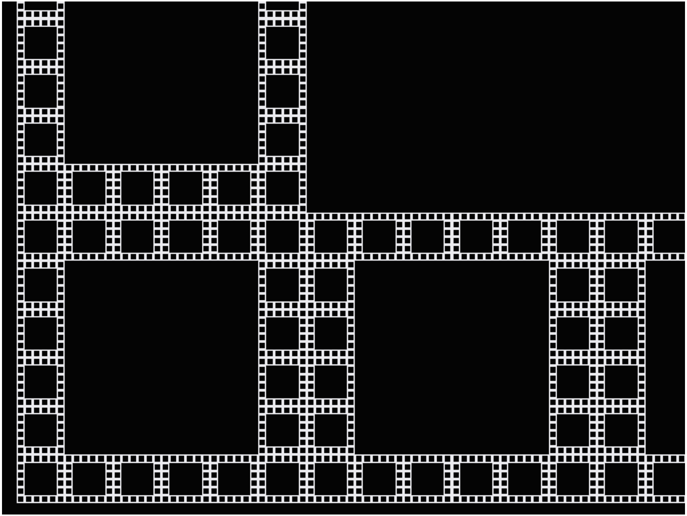




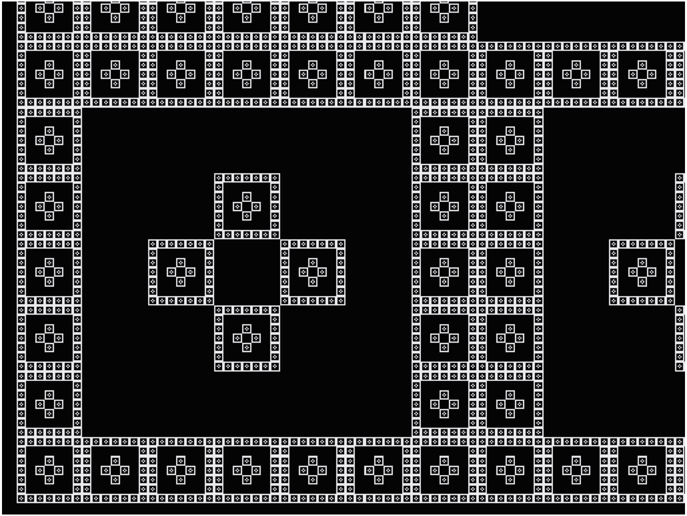
$x \ \mathrm{AND}_5^4 \ y$



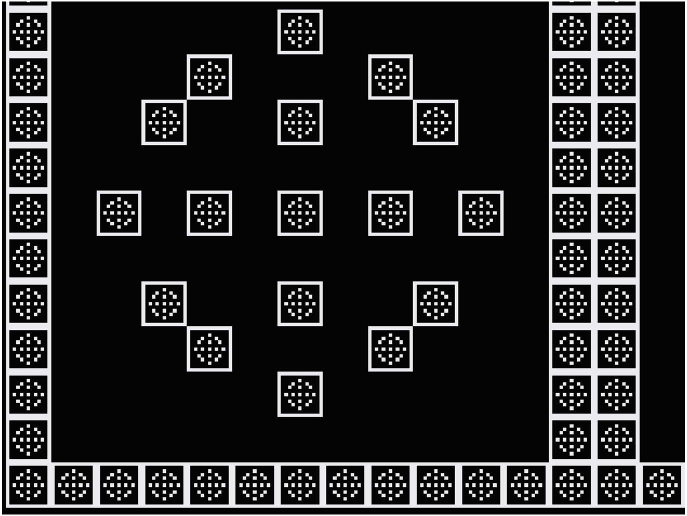
$x \text{ AND}_6^5 y$



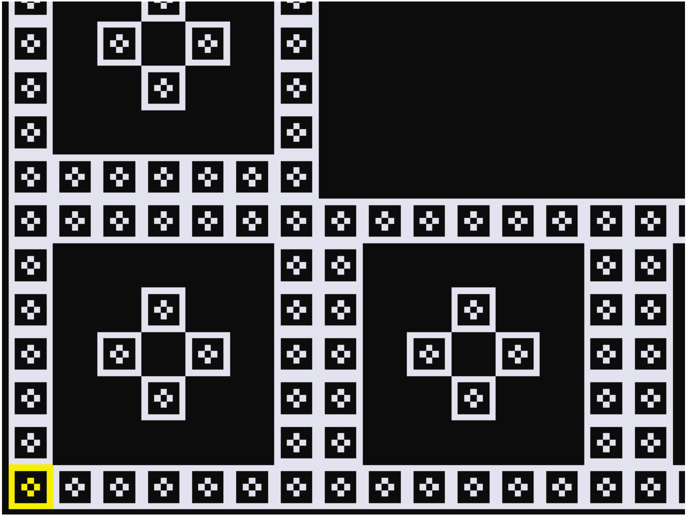
$x \text{ AND}_7^6\ y$



x AND$^{12}_{13}$ y

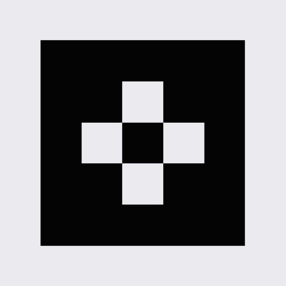


Why? Let's look at the bottom left cell (AND_7^6).



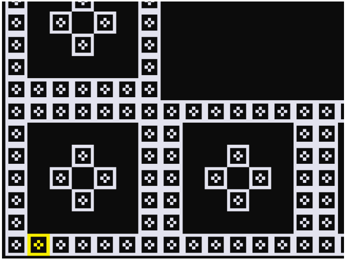
Its shape comes from the table.

	0	1	2	3	4	5	6
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**0**	1	2	3	4	5	**0**
2	**0**	2	4	**0**	2	4	**0**
3	**0**	3	**0**	3	**0**	3	**0**
4	**0**	4	2	**0**	2	4	**0**
5	**0**	5	4	3	2	1	**0**
6	**0**	**0**	**0**	**0**	**0**	**0**	**0**



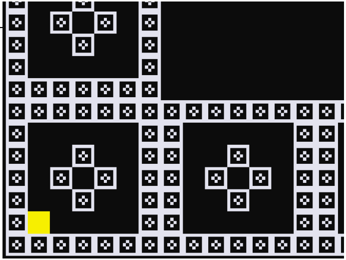
The next cell over is the same.

	10	11	12	13	14	15	16
00	**0**	**0**	**0**	**0**	**0**	**0**	**0**
01	**0**	1	2	3	4	5	**0**
02	**0**	2	4	**0**	2	4	**0**
03	**0**	3	**0**	3	**0**	3	**0**
04	**0**	4	2	**0**	2	4	**0**
05	**0**	5	4	3	2	1	**0**
06	**0**	**0**	**0**	**0**	**0**	**0**	**0**



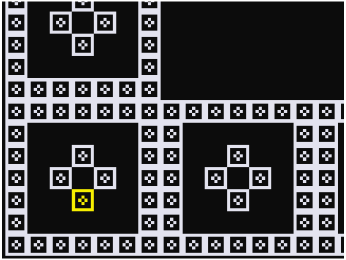
The next cell up is different.

	10	11	12	13	14	15	16
10	$\neq 0$	...					
11	...						
12	...						
13	...						
14	...						
15	...						
16	...						



What about the middle?

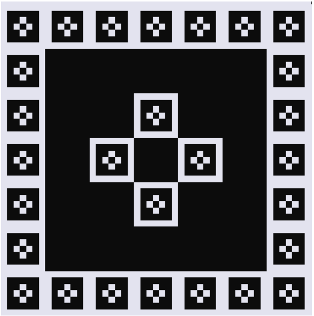
	30	31	32	33	34	35	36
20	**0**	**0**	**0**	**0**	**0**	**0**	**0**
21	**0**	1	2	3	4	5	**0**
22	**0**	2	4	**0**	2	4	**0**
23	**0**	3	**0**	3	**0**	3	**0**
24	**0**	4	2	**0**	2	4	**0**
25	**0**	5	4	3	2	1	**0**
26	**0**	**0**	**0**	**0**	**0**	**0**	**0**



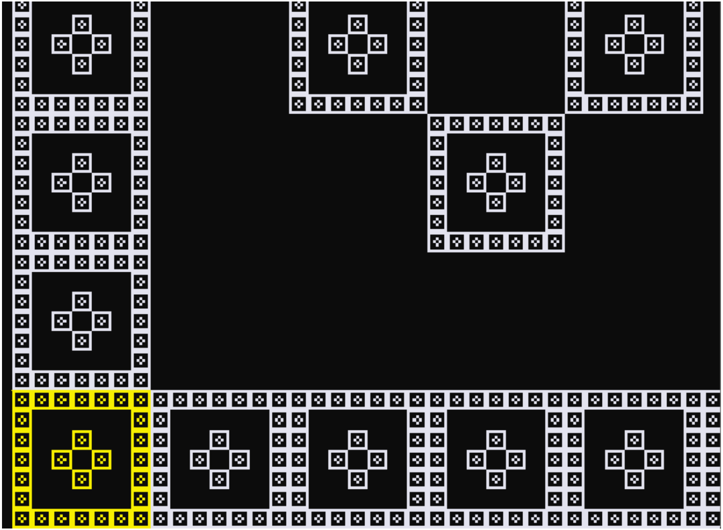
The pattern also appears in the 30×20, 40×30, and 30×40 blocks because for those the product of the left digits is 0.

The whole 2-digit block...

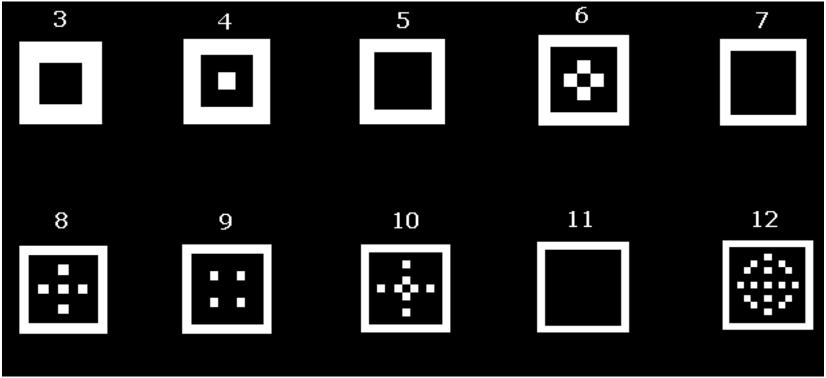
	0	1	2	...	64	65	66
0							
1							
2							
...							
64							
65							
66							



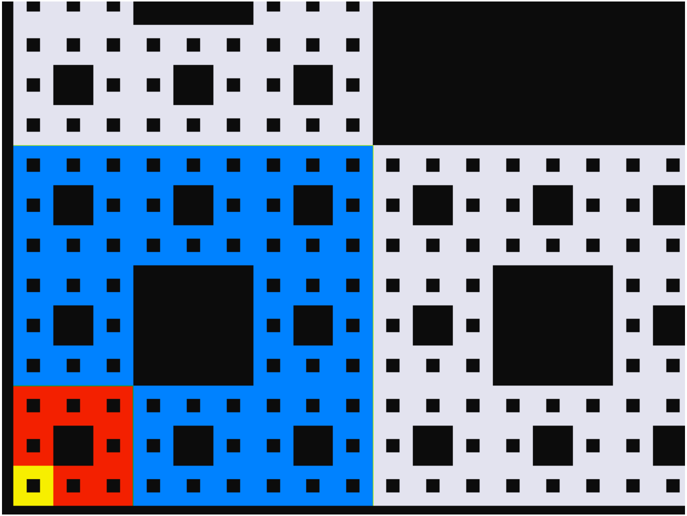
. . . is repeated.



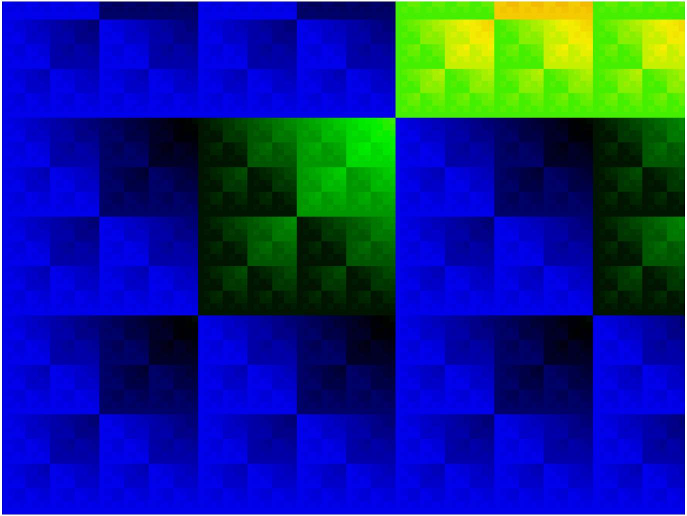
Patterns for Various Bases



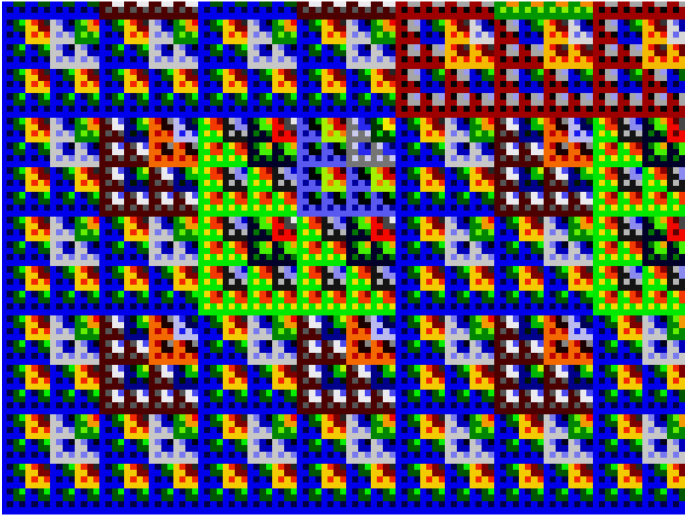
x AND$_3^2$ y Again



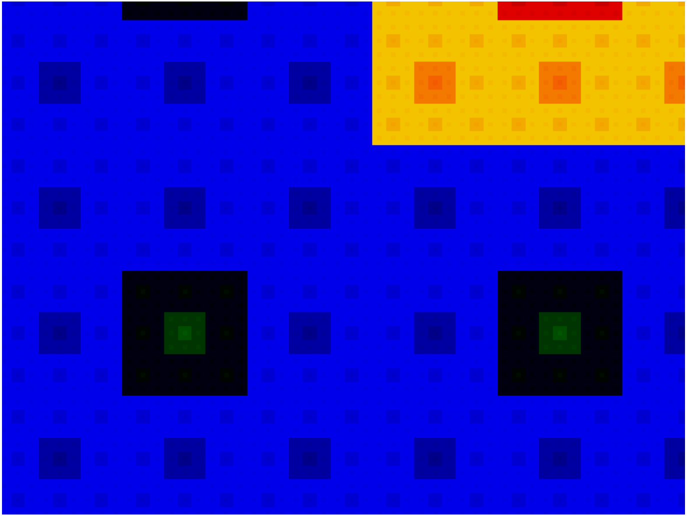
x AND y in Color



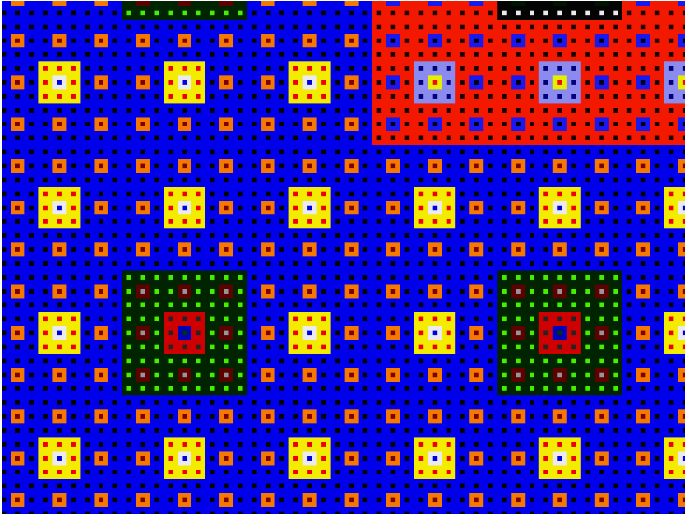
x AND y in Color (different coloring scheme)



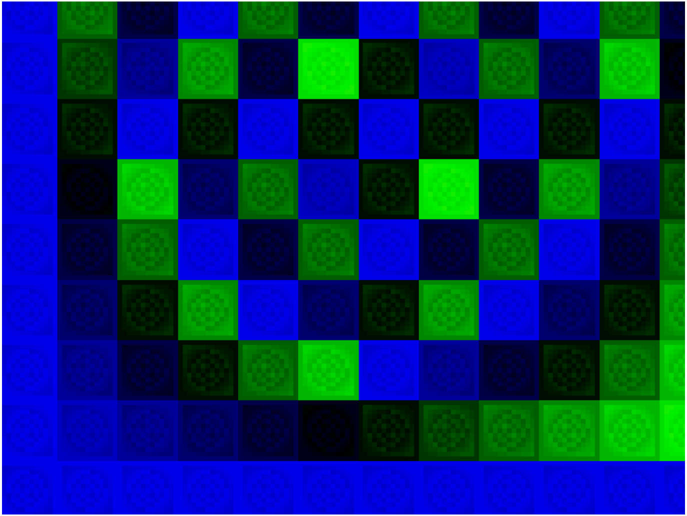
x AND$^2_3$ y in Color



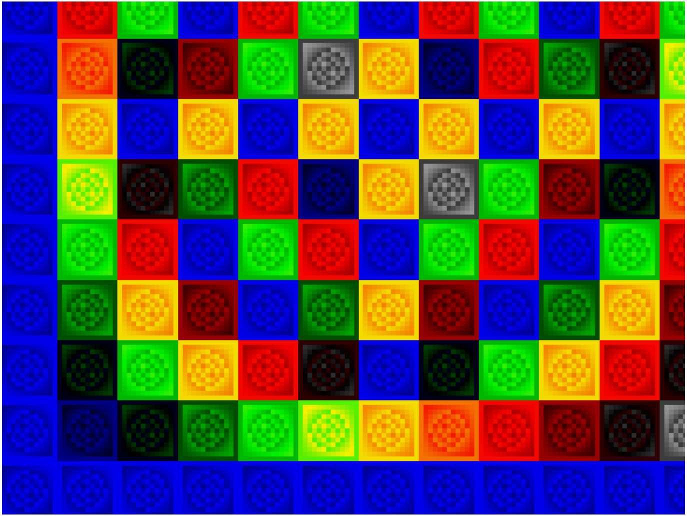
x AND$_3^2$ y in Color (different coloring scheme)



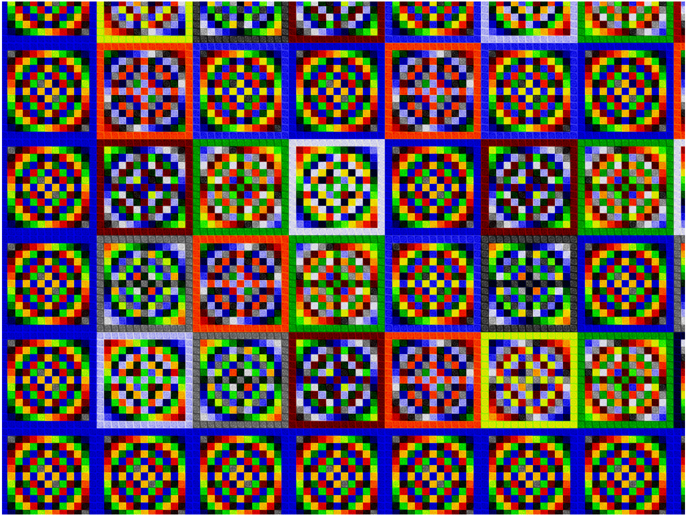
x AND$^{12}_{13}$ y in Color



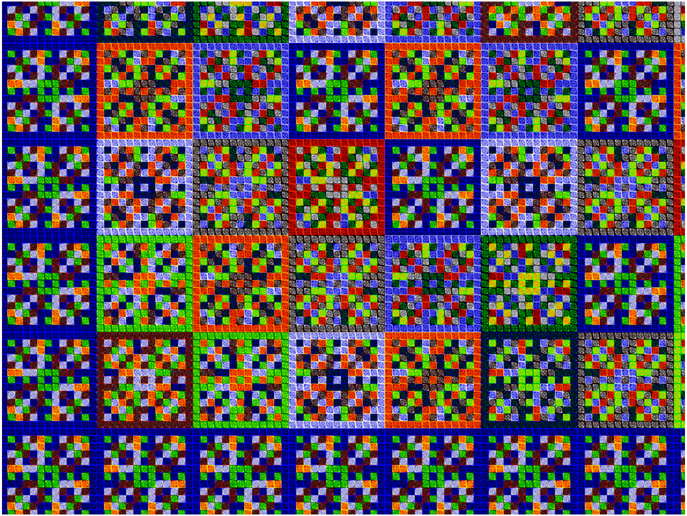
x AND$^{12}_{13}$ y in Color (different coloring scheme)



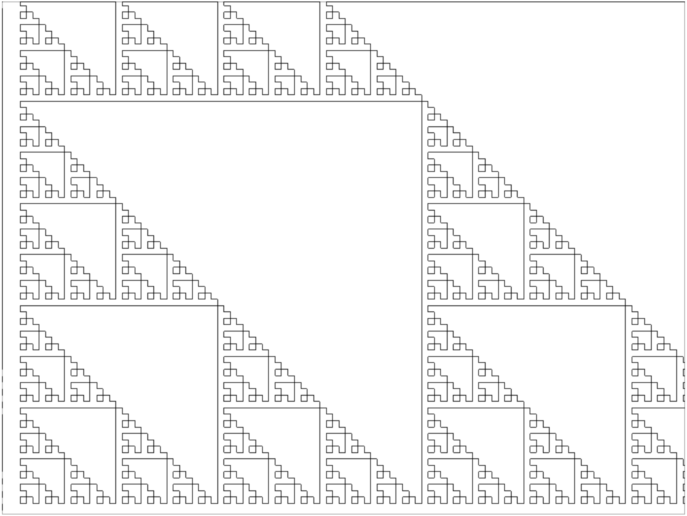
x AND$_{13}^{12}$ y in Color (another coloring scheme)



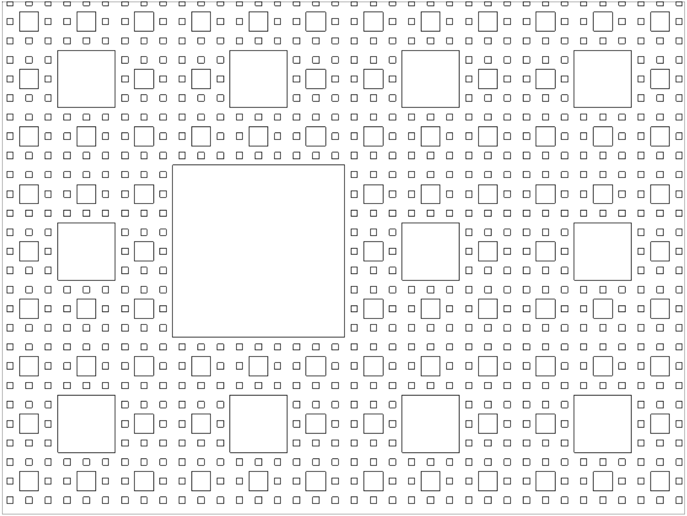
x AND$^{12}_{13}$ y in Color (yet another coloring scheme)



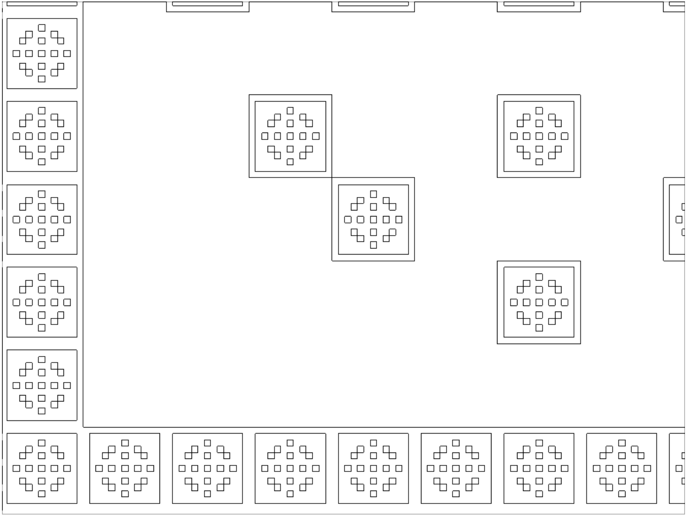
x AND y from Implicit Plotter



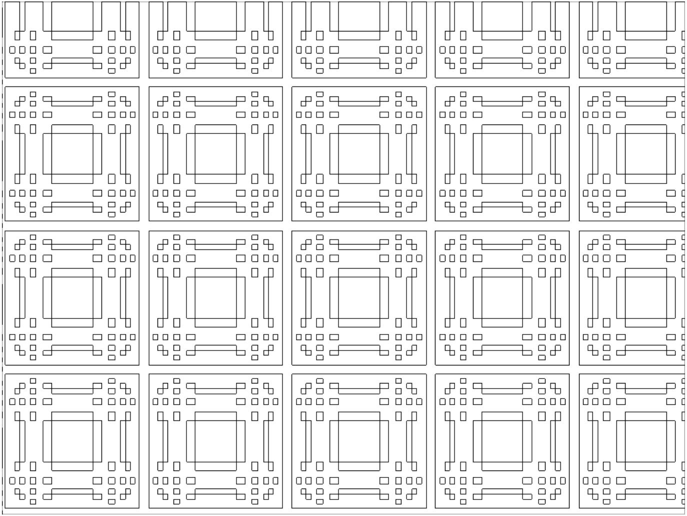
x AND$_3^2$ y from Implicit Plotter



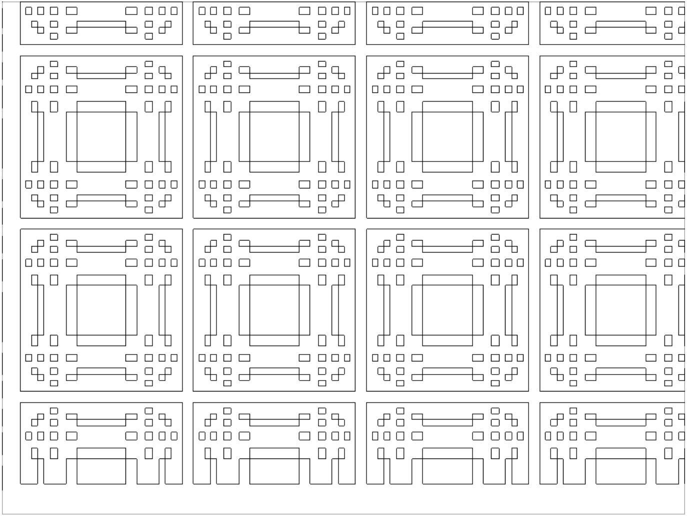
$x\ \mathrm{AND}_{13}^{12}\ y$ from Implicit Plotter



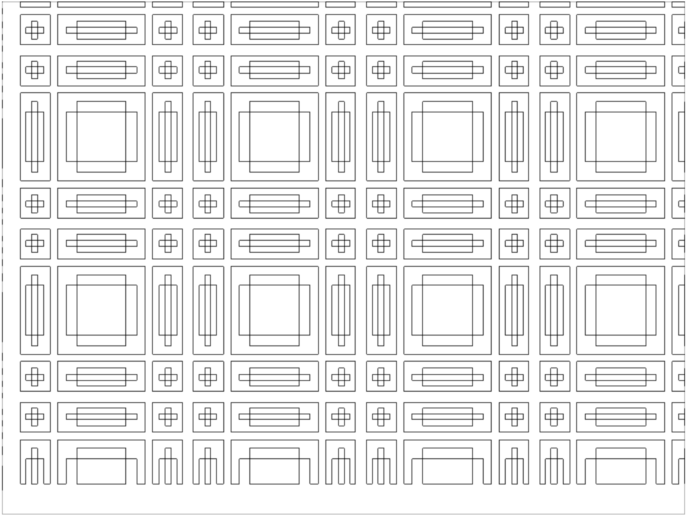
$10\sin(x)\ \mathrm{AND}^{12}_{13}\ 10\sin(y)$



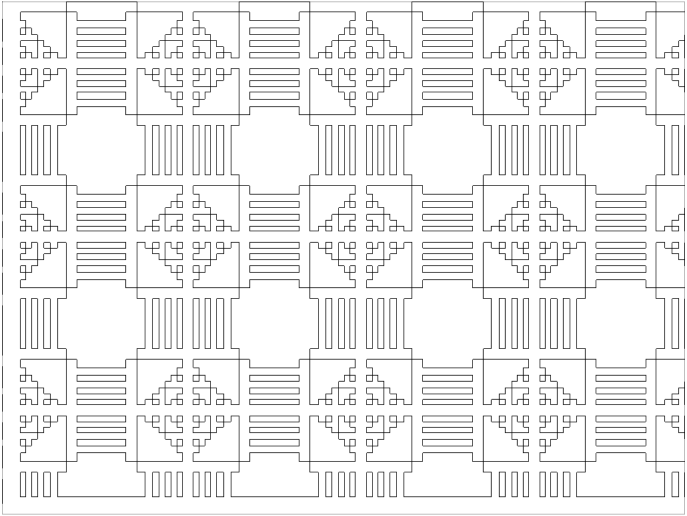
$10\sin(x)\ \mathrm{AND}_{13}^{12}\ 10\sin(y)$



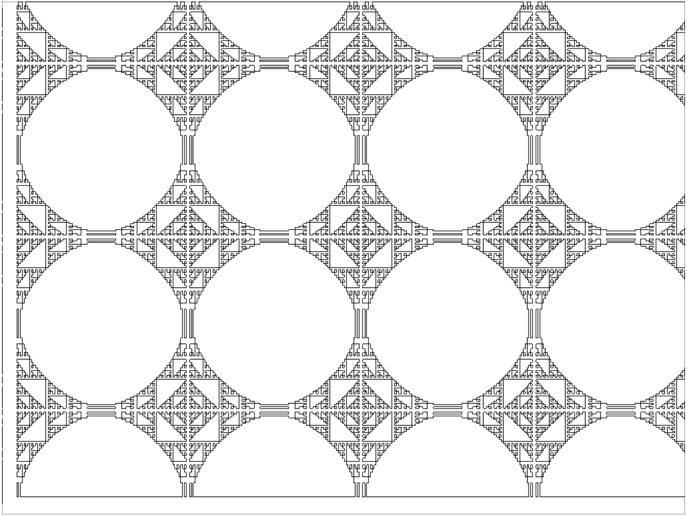
$10\sin(x)\ \mathrm{AND}_{13}^{12}\ 10\cos(y)$



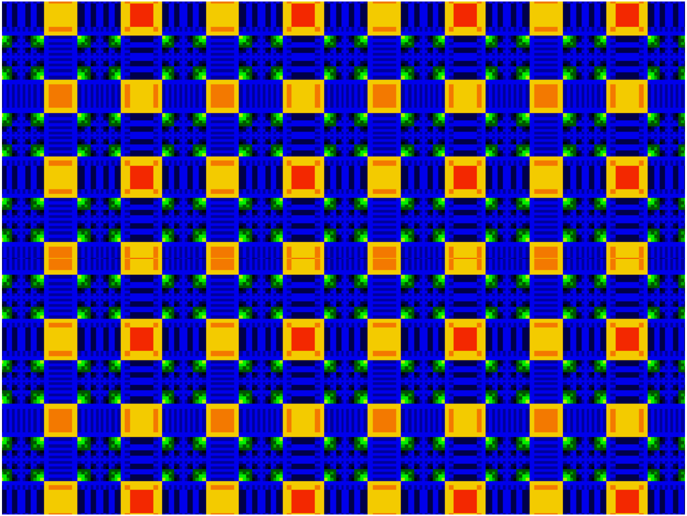
$10\sin(x)$ AND $10\cos(y)$



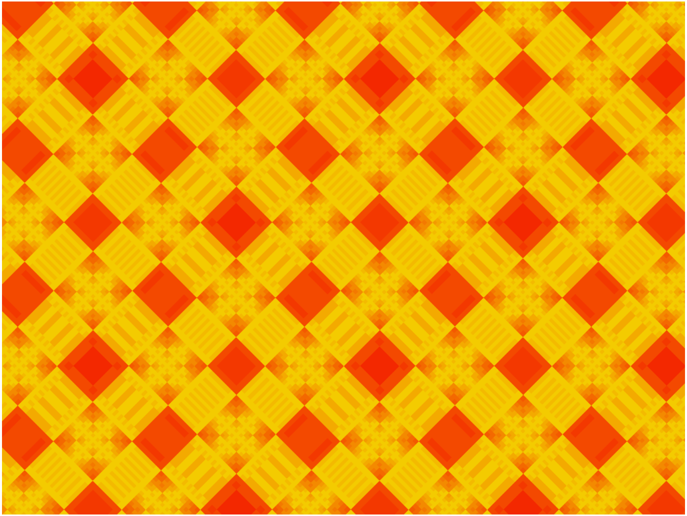
$30\sin(x)$ AND $30\sin(y)$



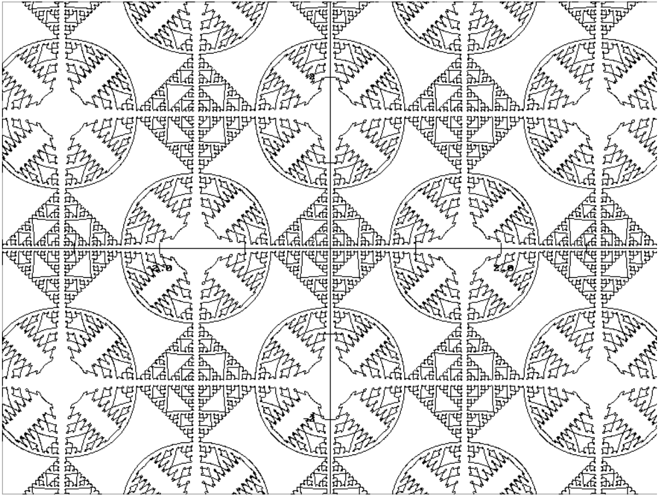
$10\sin(x)$ AND $10\cos(y)$



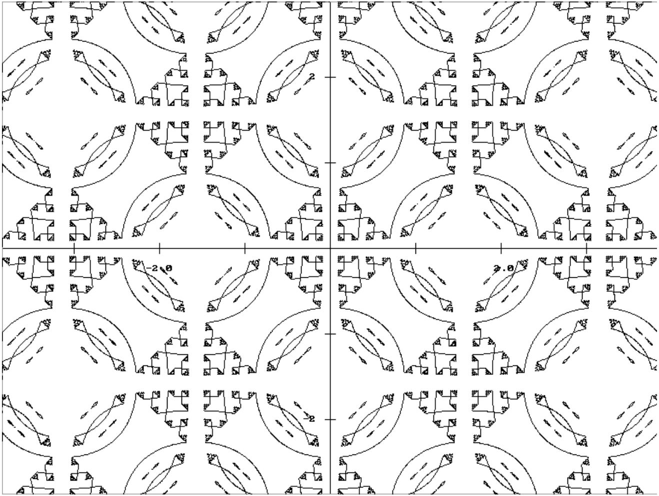
$10\sin(y+x)$ AND $10\cos(y-x)$



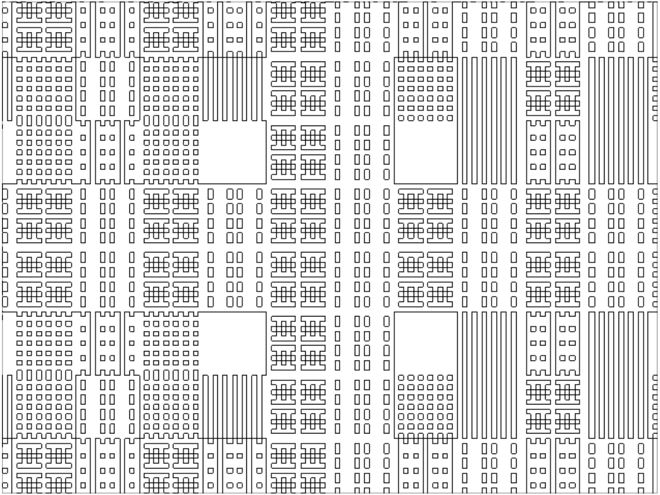
$100\sin(x)\cos(y)$ AND $100\cos(x)\sin(y) = 2$



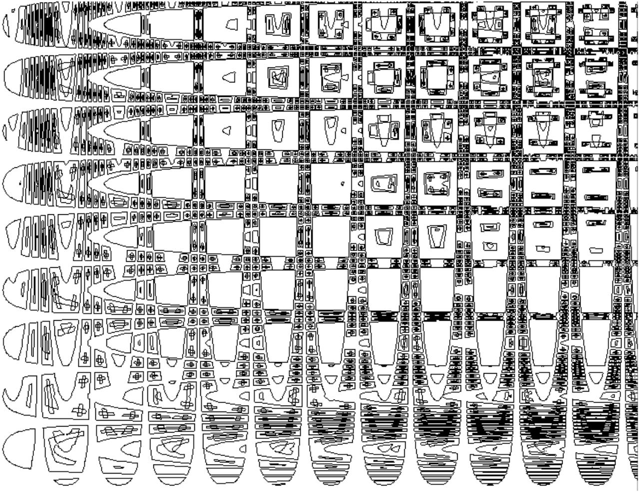
$$100 \sin(x) \cos(y) \ \mathrm{AND}_3^2 \ 100 \cos(x) \sin(y) = 8$$



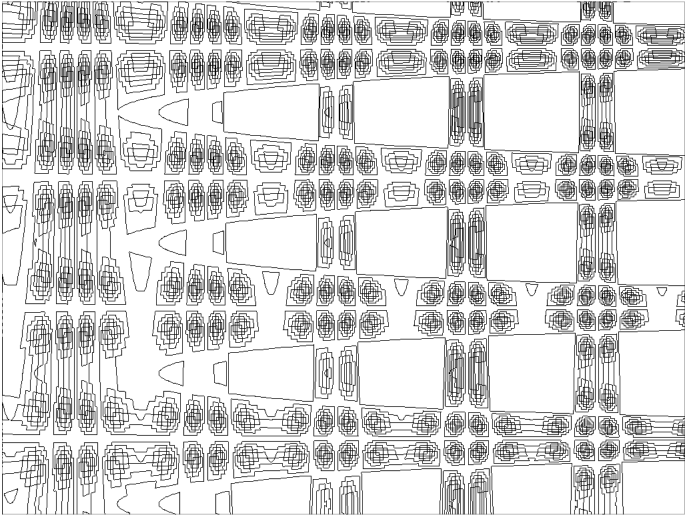
$x \text{ AND}^{13}_{2+(x \bmod 2)} y$



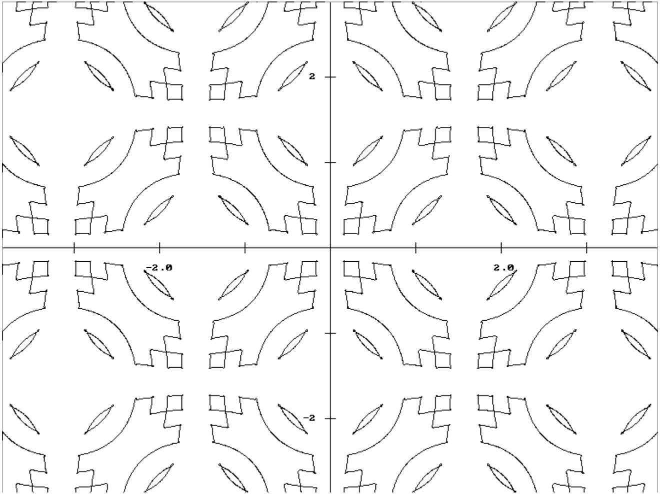
$x \cos(y)$ AND$_7^6$ $y \sin(x)$



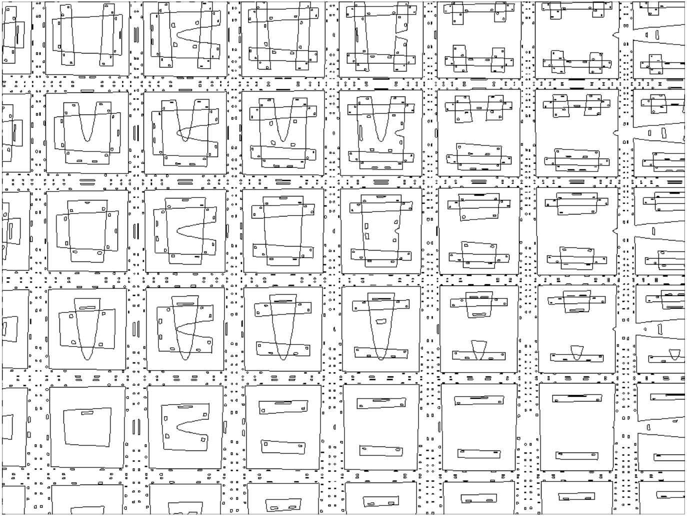
$x\cos(y)\ \mathrm{AND}_{13}^{12}\ y\cos(x) = 4$



$100\sin(x)\cos(y)\ \mathrm{AND}_{13}^{12}\ 100\cos(x)\sin(y) = 14$



$x\cos(y)\ \mathrm{AND}_7^6\ y\sin(x) = 4$



$x\cos(y)$ AND_7^6 $y\sin(x)$