Iterating a discontinuous function

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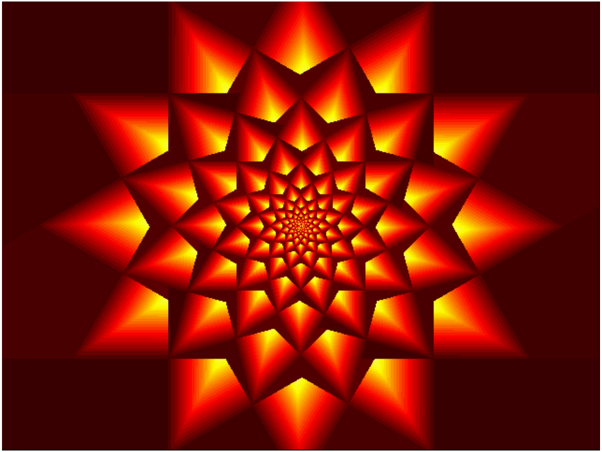
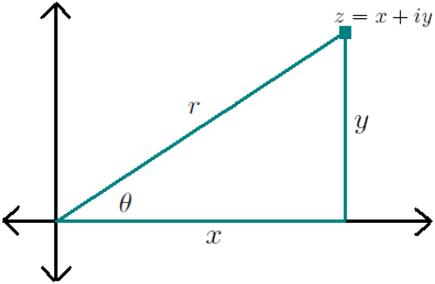


Complex numbers

$x + iy \longleftrightarrow re^{i\theta}$

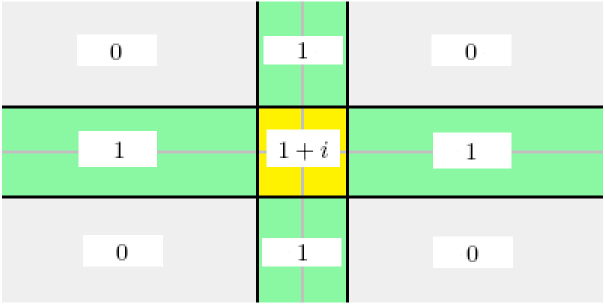


$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$\Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ (a rotation and a dilation)

The function

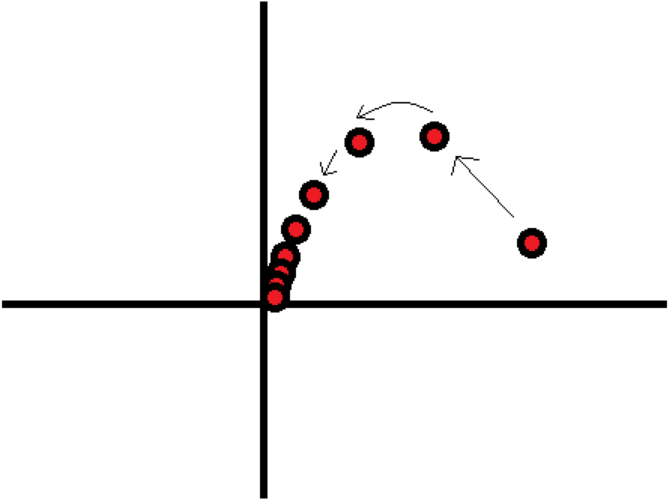
$f(z) = c\gamma(z)z$, where $\gamma(z)$ is defined as below.



In the outside strips $f(z)$ rotates points by $\arg(c)$ and dilates them by $|c|$.
In the inner square, $f(z)$ rotates points by $\arg(c) + 45^\circ$ and dilates them by $\sqrt{2}|c|$.

Iteration

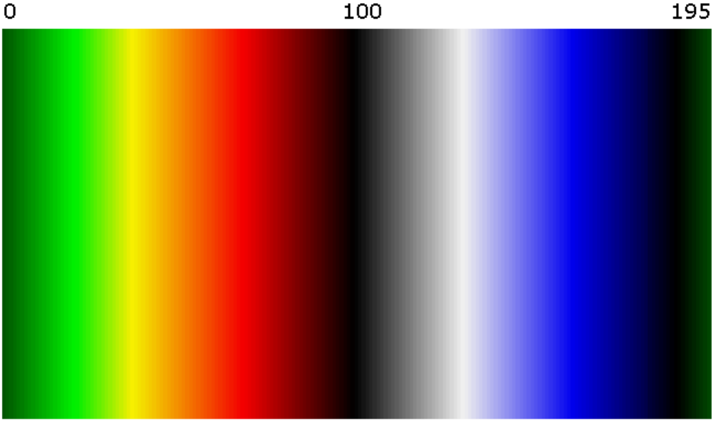
Plug $z = x + iy$ into $f(z)$. Get a value, and plug that value into the function. Then plug the result of that into the function, etc.



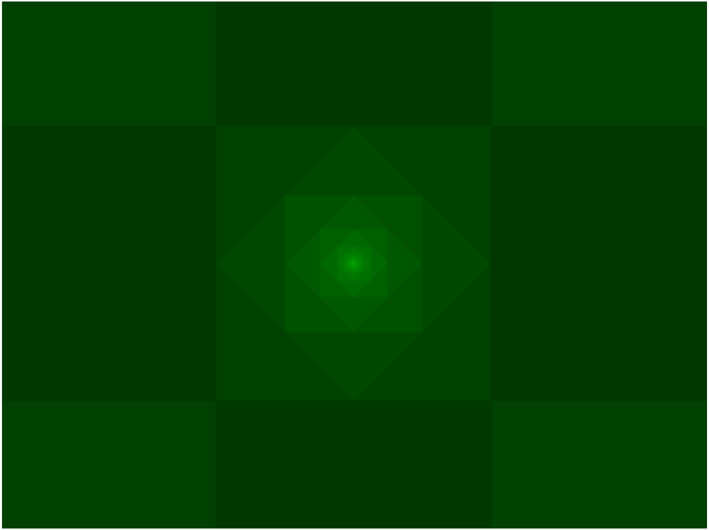
Iteration

- Do this for each point in the plane. Keep iterating until the difference between successive iterates is less than 10^{-5} or the real and imaginary parts of z leave $[-10^5, 10^5]$ (converge to ∞).
- Color the point (x, y) according to how many iterations it takes until we stop.
- It is possible that the iteration may never stop. Give up after a few hundred iterations and color the point yellow.

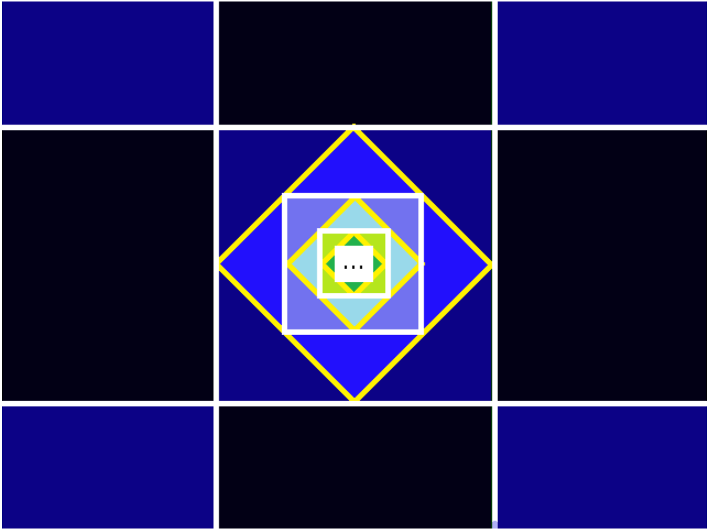
Coloring



$f(z) = z\gamma(z)$



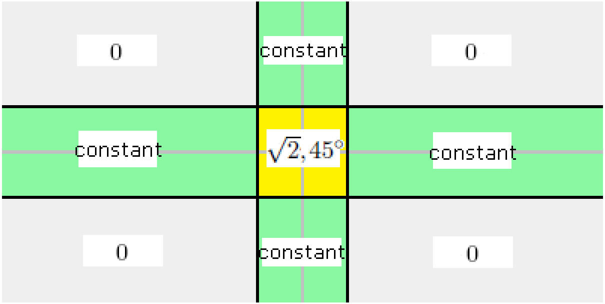
$f(z) = z\gamma(z)$



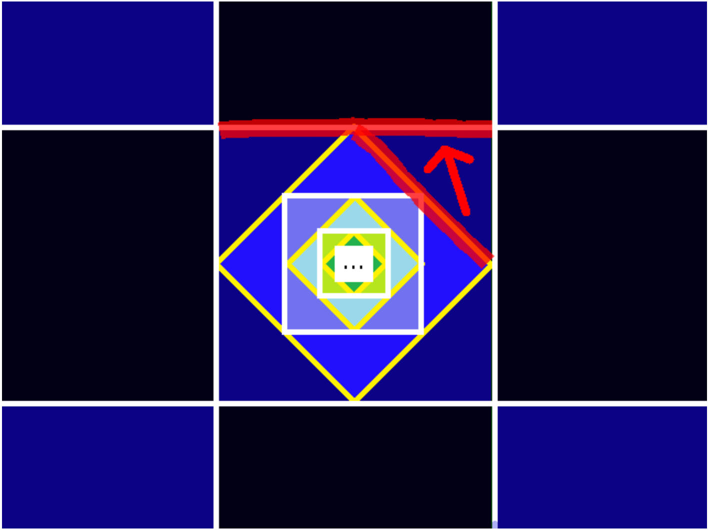
$f(z) = z\gamma(z)$

Given $z = re^{i\theta}$, $f(z)$ is described by

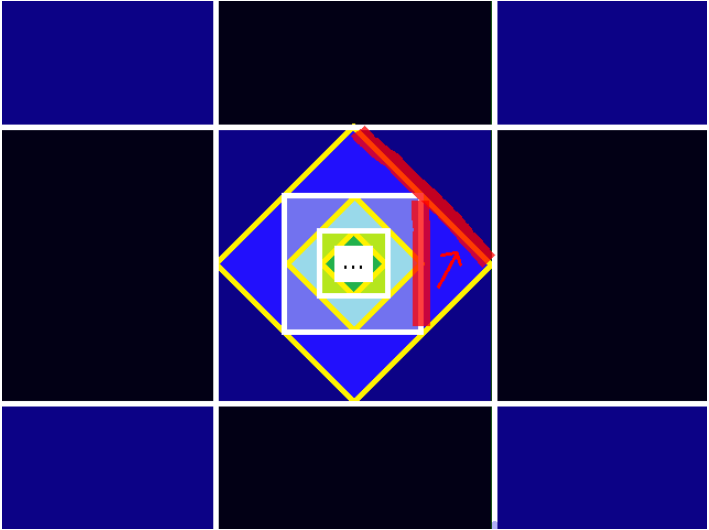
$$\begin{cases} r \mapsto \sqrt{2}r, \quad \theta \mapsto \theta + 45^\circ & \text{center box} \\ r, \theta \text{ constant} & \text{strips} \\ r, \theta \mapsto 0 & \text{elsewhere} \end{cases}$$



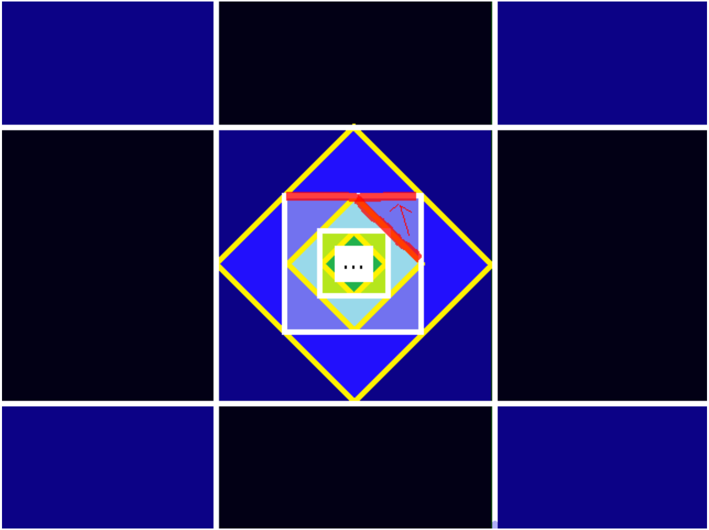
$f(z) = z\gamma(z)$



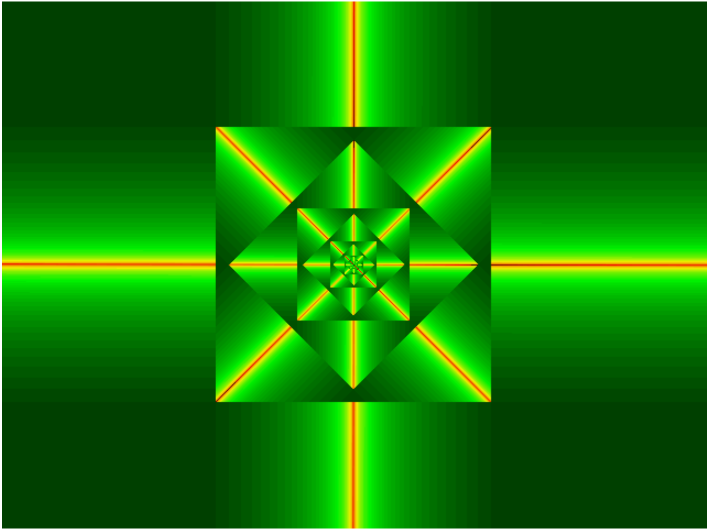
$f(z) = z\gamma(z)$



$f(z) = z\gamma(z)$



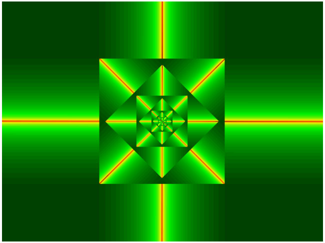
$c = 1.1$



$c = 1.1$

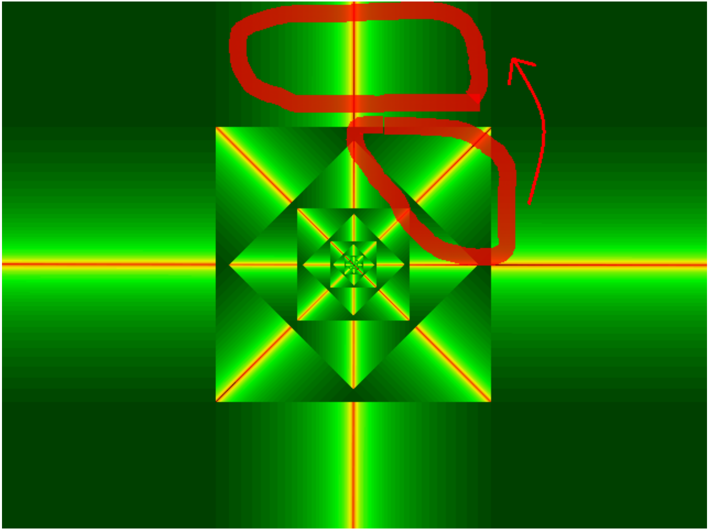
$f(z)$ is described by:

$$\begin{cases} (1.1\sqrt{2}, \quad 45^\circ) & \text{center box} \\ (1.1, \quad 0^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



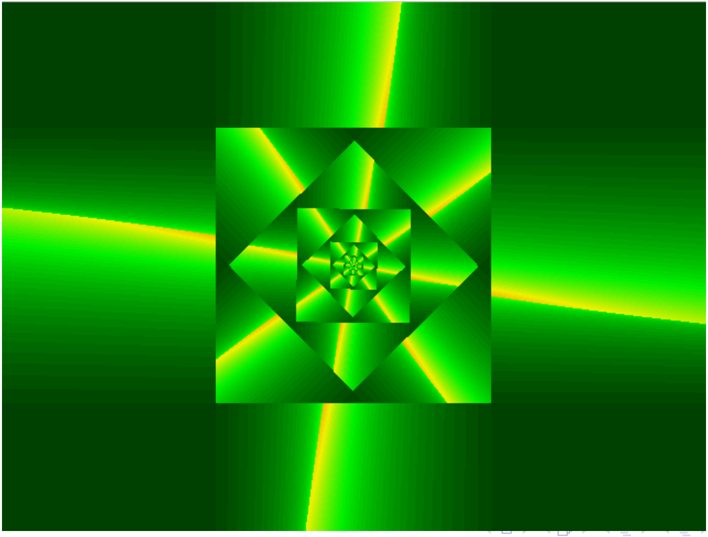
In the outside strips, the small dilation leads to slow convergence. Points within the square eventually get pushed into the outside strips.

$c = 1.1$

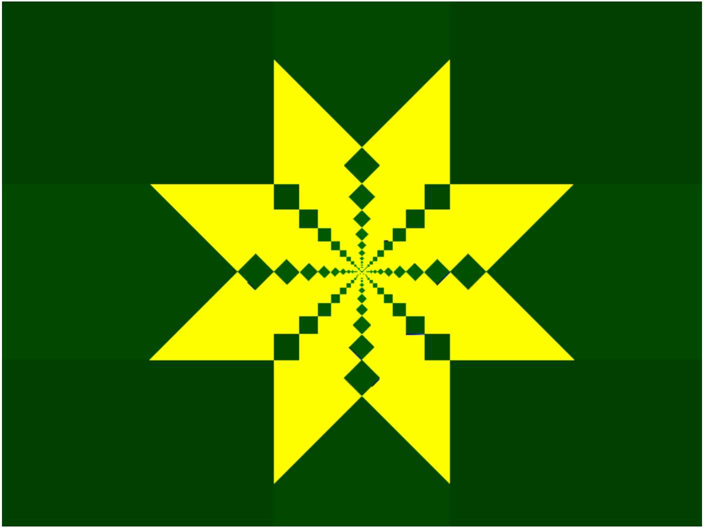


$c = 1.1 + .01i$

Adding a small imaginary term adds a bit of rotation, but no major change.

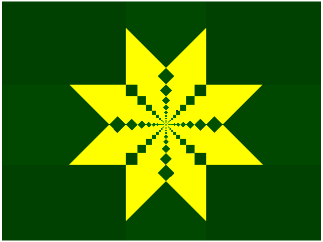


$c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$



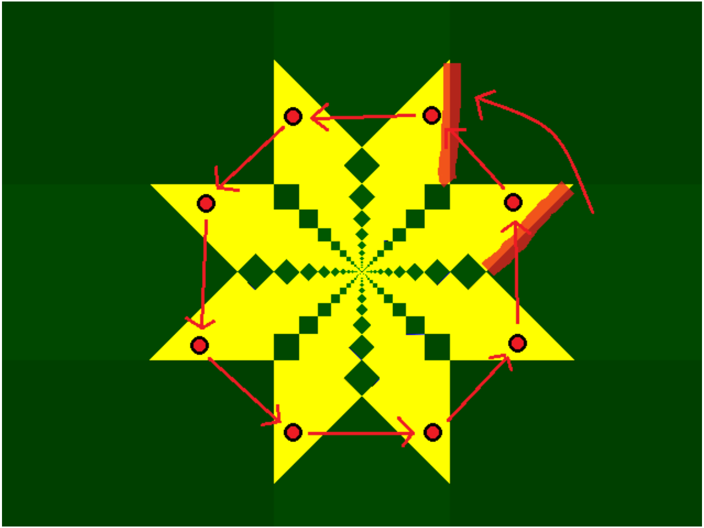
$f(z)$ is described by

$$\begin{cases} (\sqrt{2}, \quad 90^\circ) & \text{center box} \\ (1, \quad 45^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



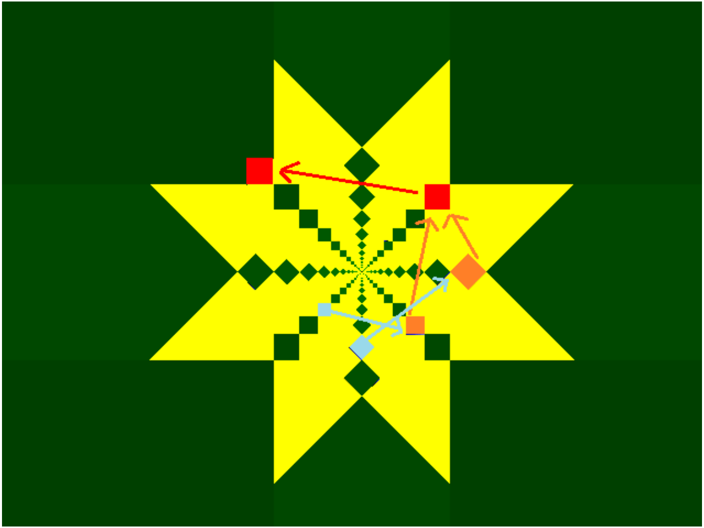
$c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Many points will cycle endlessly.



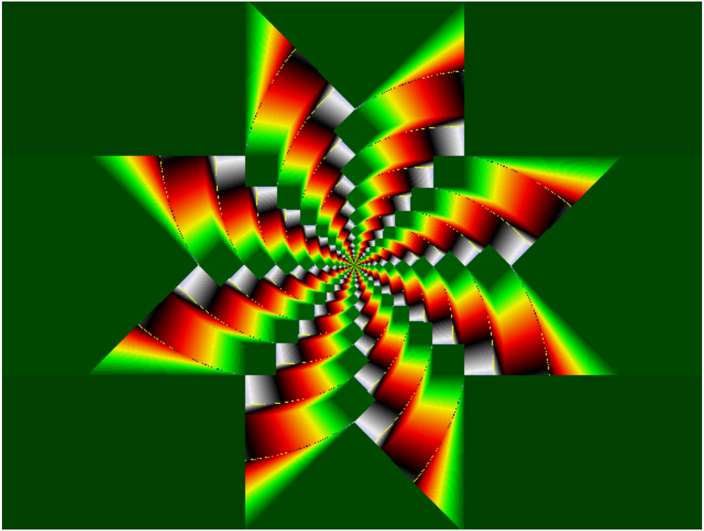
$c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

Where the green boxes and diamonds come from:



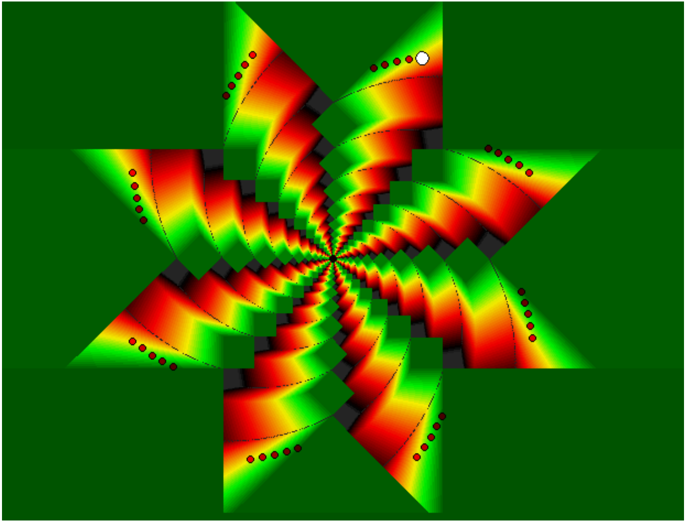
$c = .700 + .709i$

Move from $c \approx .707 + .707i$ to $.700 + .709i$.

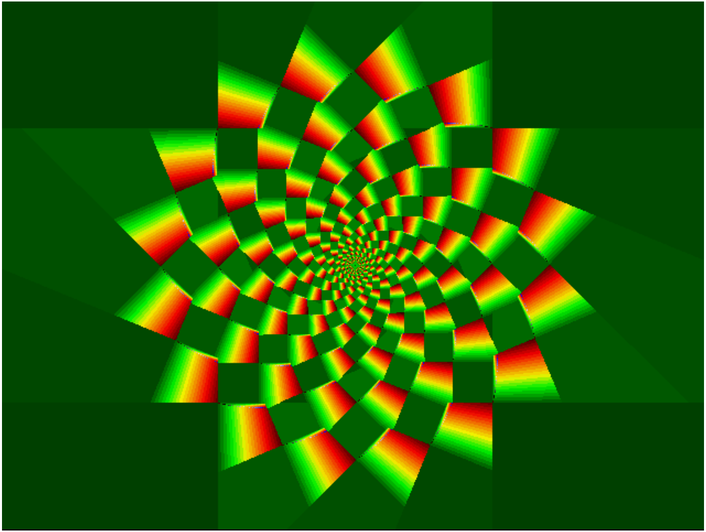


$c = .700 + .709i$

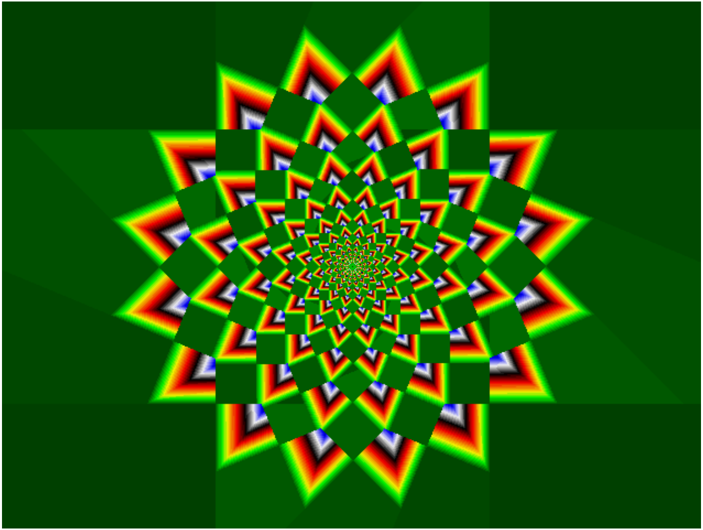
The red circles are the actual iterates. Rotation is not quite $45°$. The slight perturbation adds up.



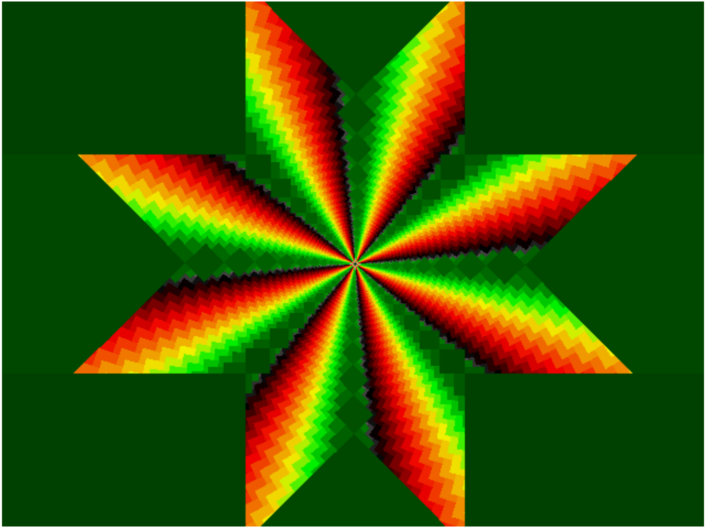
$c = .926 + .381i$



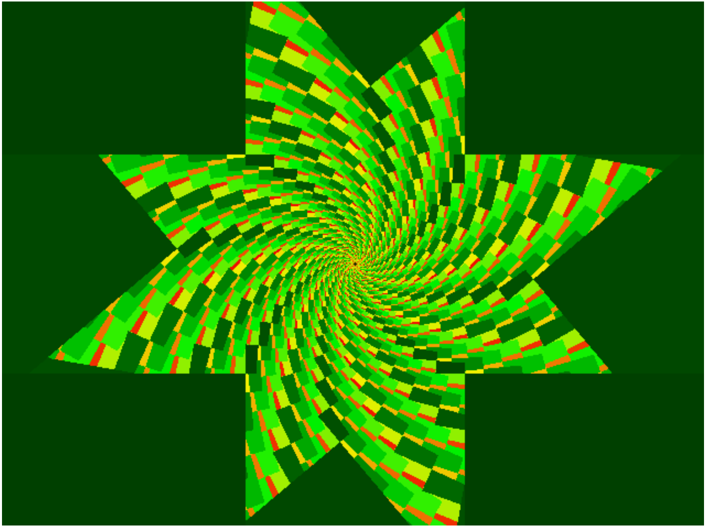
$c = .926 + .384i$



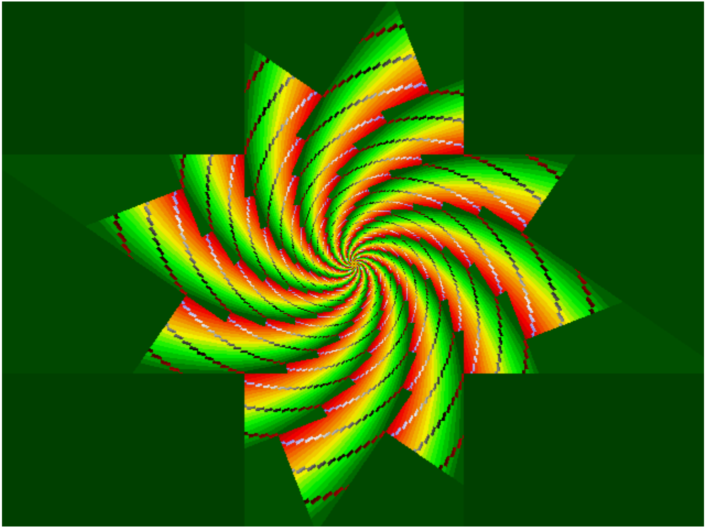
$c = .655 + .653i$



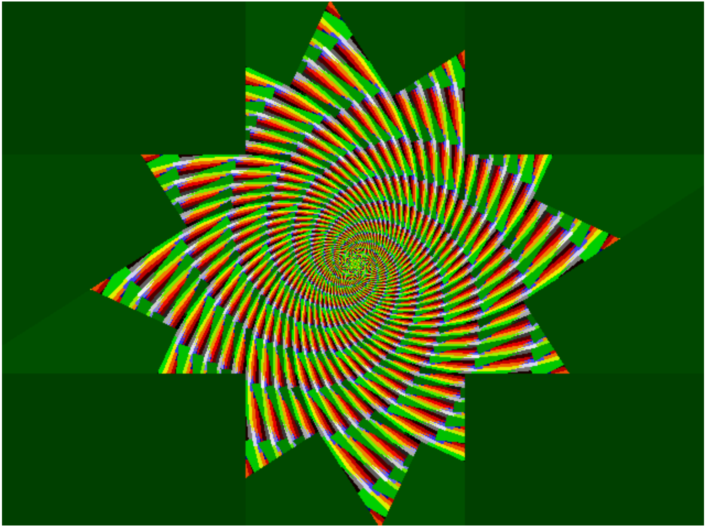
$c = .561 + .667i$



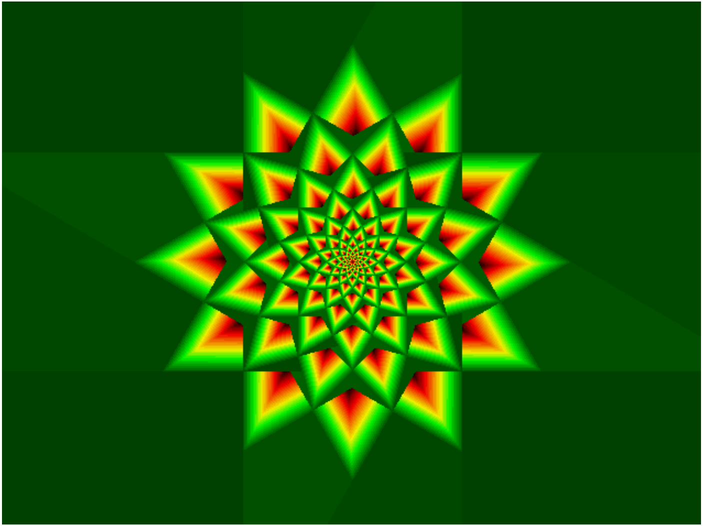
$c = .752 + .516i$



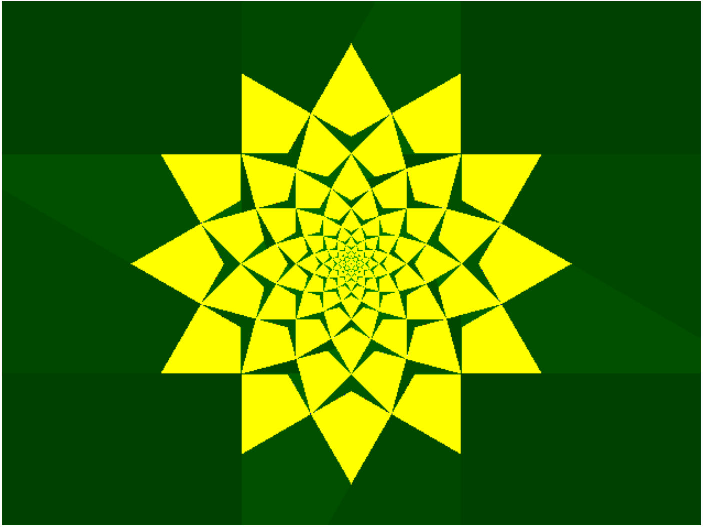
$c = .489 + .765i$



$c = .870 + .504i$



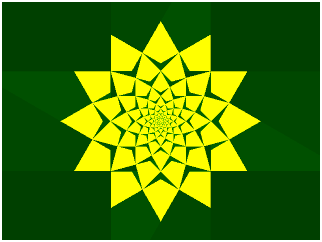
$c = \frac{\sqrt{3}}{2} + \frac{1}{2}i$



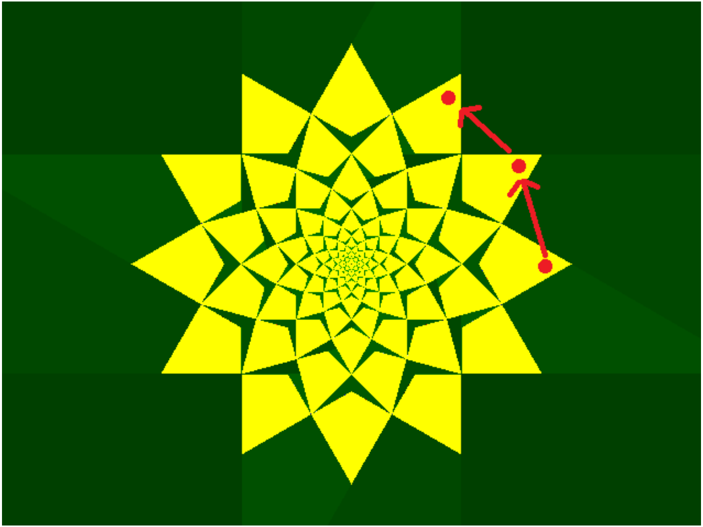
$c = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$f(z)$ is described by

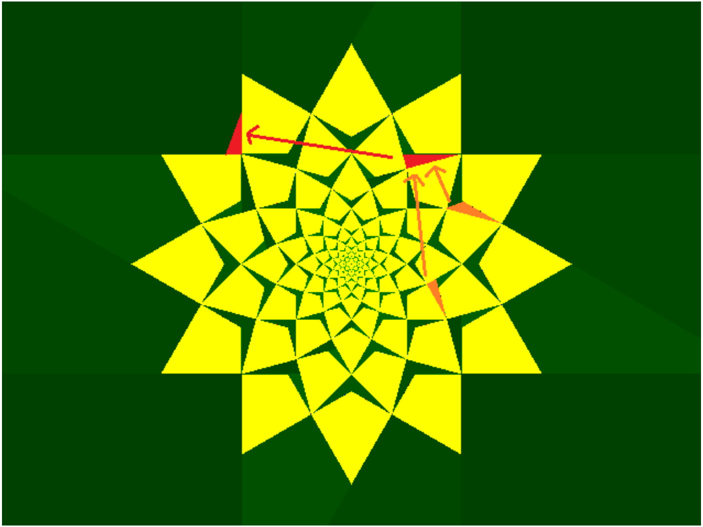
$$\begin{cases} (\sqrt{2}, \quad 75^\circ) & \text{center box} \\ (1, \quad 30^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



Get cycles again because 360 is divisible by 30.

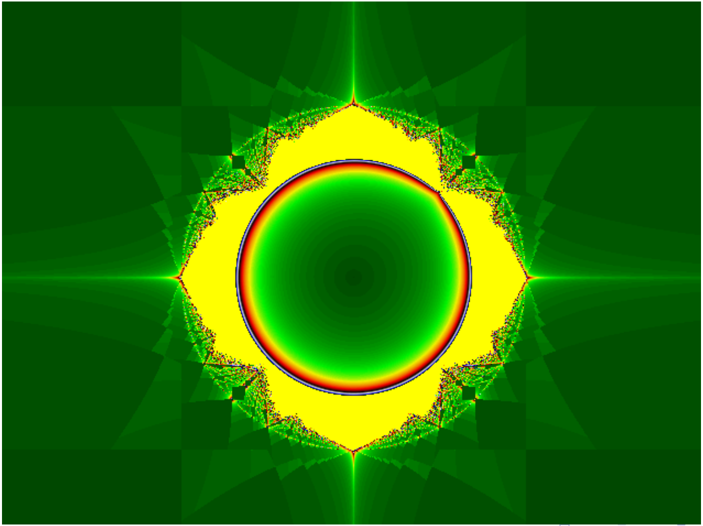


$c = \frac{\sqrt{3}}{2} + \frac{1}{2}i$



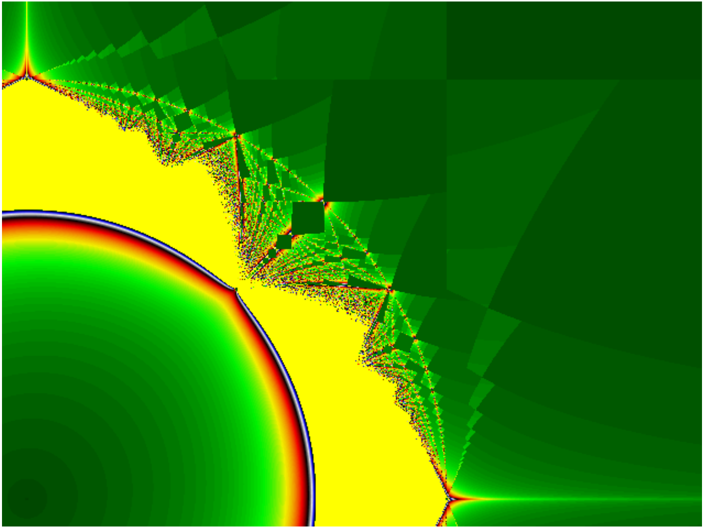
Index set

For each value of c, see what color we get when we iterate starting at $z = 1$.

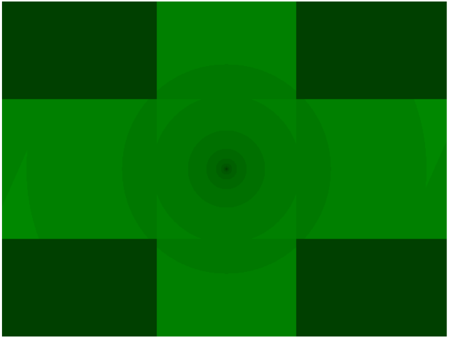


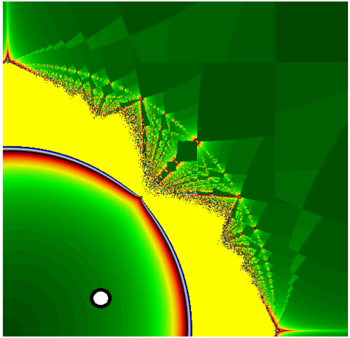
Index set close-up

Color of $z = 1$ is somewhat representative of the entire image.

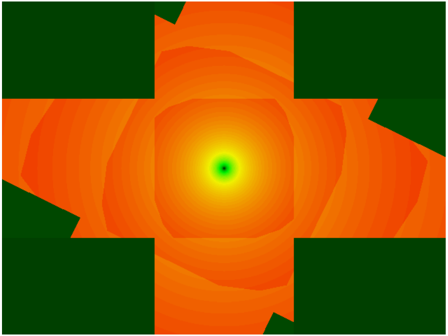


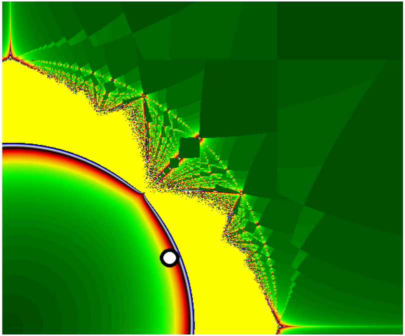
$c = .337 + .151i$

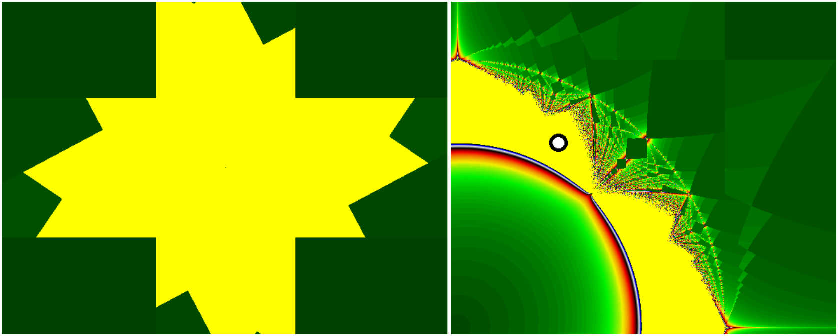




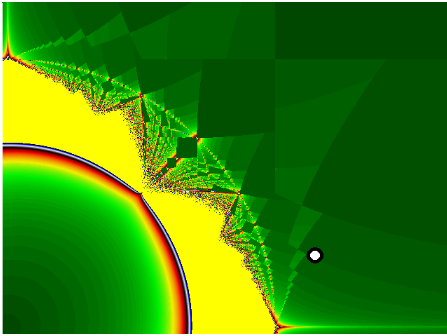
$c = .584 + .287i$



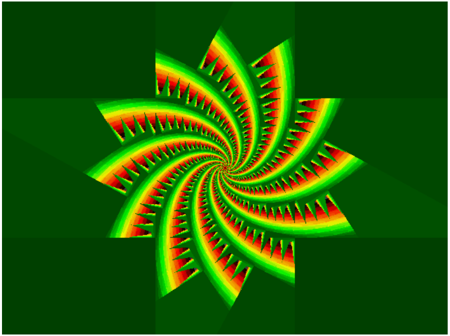


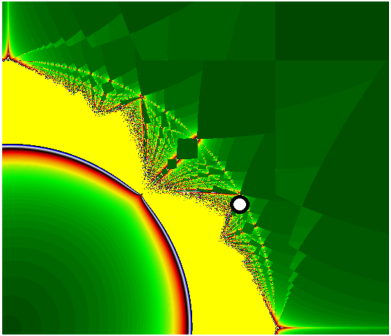


$c = .1139 + .271i$



$c = .854 + .465i$





Things to try

- Other piecewise functions
- Change z to z^2 or something else
- Other types of transformations