Iterating the complex logarithm

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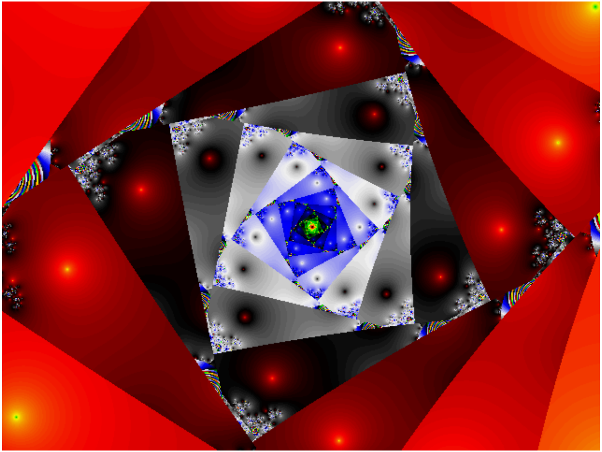


Complex numbers

$i = \sqrt{-1}$ (solution to $x^2 + 1 = 0$)

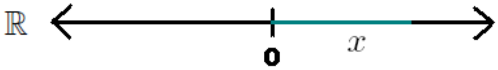
Examples: $2i, \quad 3 + 4i, \quad -.2 + .76i$

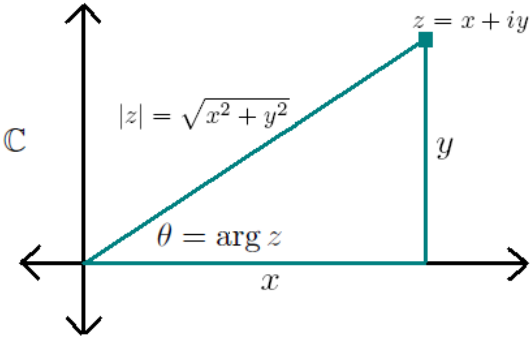
Addition: $(2 + 3i) + (5 + 8i) = 7 + 11i$

Multiplication: $(2 + 3i)(5 + 8i) = 10 + 31i + 24i^2 = -14 + 31i$

Division: $\frac{2+3i}{5+8i} = \frac{2+3i}{5+8i} \cdot \frac{5-8i}{5-8i} = \frac{34-8i}{89} = \frac{34}{89} + \frac{8}{89}i$

Picturing them





Iteration

Example: Let $f(x) = x^2$ and start with $x = 2$.

$f(2) = 4$
$f(4) = 16$
$f(16) = 256$
$f(256) = 65536$
$\ldots$

Iterates are approaching ∞.

A different starting point

Let $f(x) = x^2$ and start with $x = \frac{1}{2}$.

$f(\frac{1}{2}) = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{16}$

$f(\frac{1}{16}) = \frac{1}{256}$

$f(\frac{1}{256}) = \frac{1}{65536}$

$\ldots$

Iterates are approaching 0.

Another example

Let $f(x) = -x$ and start with $x = 1$.

$f(1) = -1$
$f(-1) = 1$
$f(1) = -1$
$f(-1) = 1$
$\ldots$

Iterates are not settling down on a value.

Coloring by convergence

Color each point according to how fast it converges.

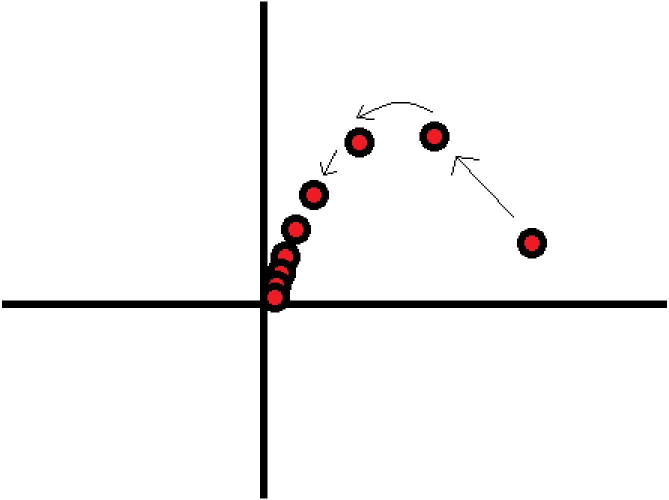
Count how many iterations until two successive values are within .00001 of each other.

Assign each count a color.

Convergence to infinity is still convergence (color by # of steps to exceed $\pm 10^5$).

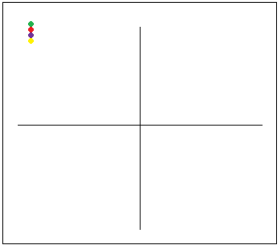
Iteration with complex numbers

Plug $z = x + iy$ into $f(z)$. Get a value, and plug that value into the function. Then plug the result of that into the function, etc.



The process

Look at all the possible starting values in a region.

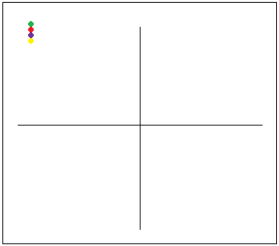


For each starting point, iterate the function.

If two successive values are within .00001 of each other, there's a very good chance that the iterates will converge.

The process, continued

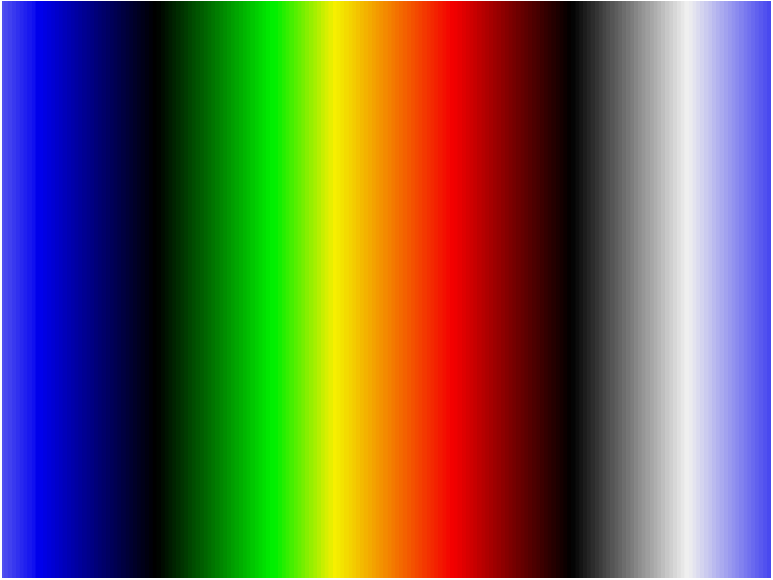
In this case, color the point with a color representing how long it took for this to happen.



It is possible that the iteration may never stop. Give up after a few hundred iterations and color the point yellow.

Note: convergence to infinity is still convergence (color by how many steps for iteration to exceed $\pm 10^5$).

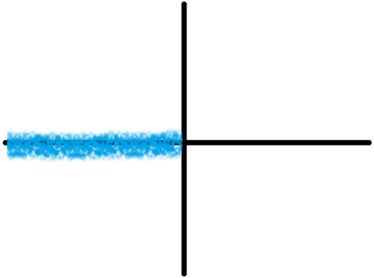
Color scheme

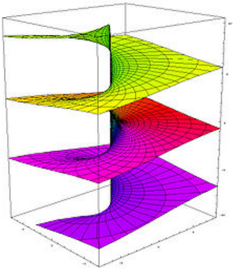


The complex logarithm

$\ln z = \ln|z| + i \arg z$

Take branch where $-\pi < \arg z \leq \pi$.

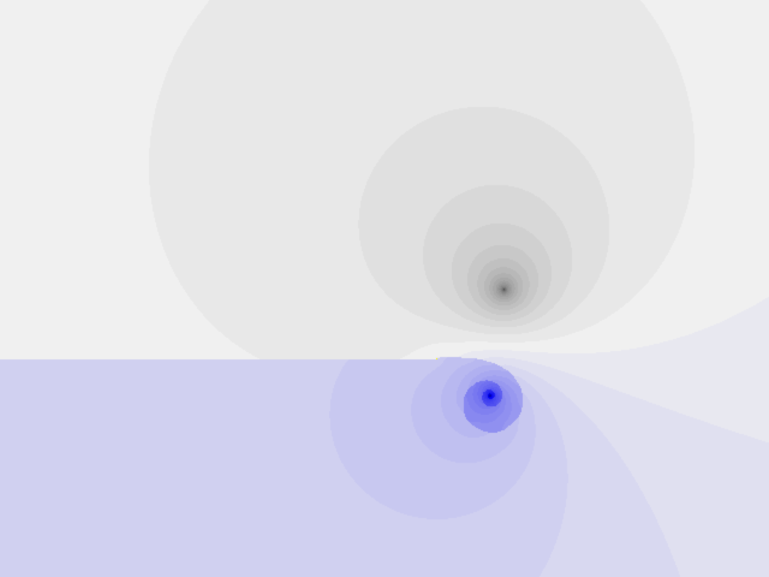




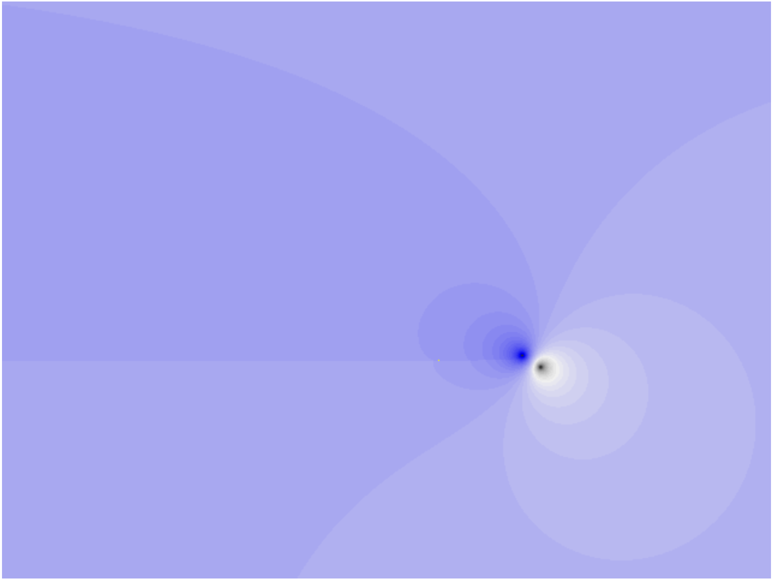
From http://www.answers.com/topic/branch-point

Will iterate $f(z) = c \ln z$ for various values of c

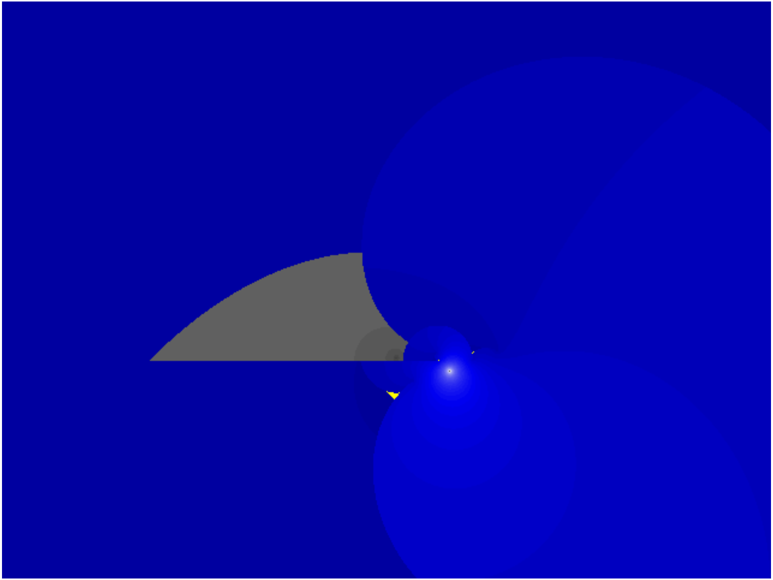
A tour of c values, $c = 2.14 + .32i$



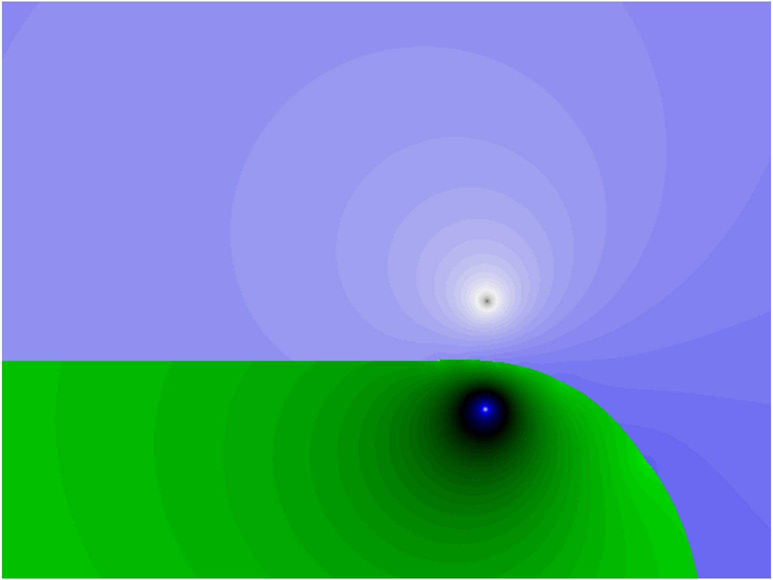
A tour of c values, $c = 2.73 + -.02i$



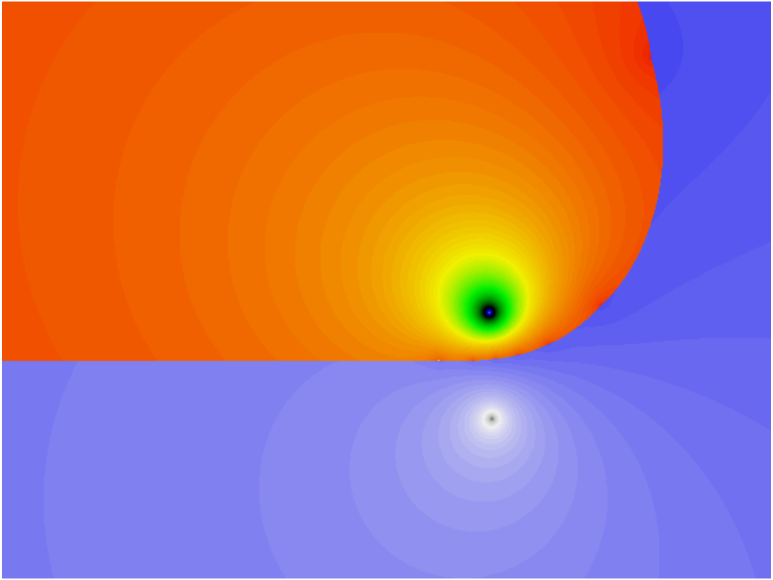
A tour of c values, $c = -.01 + .04i$



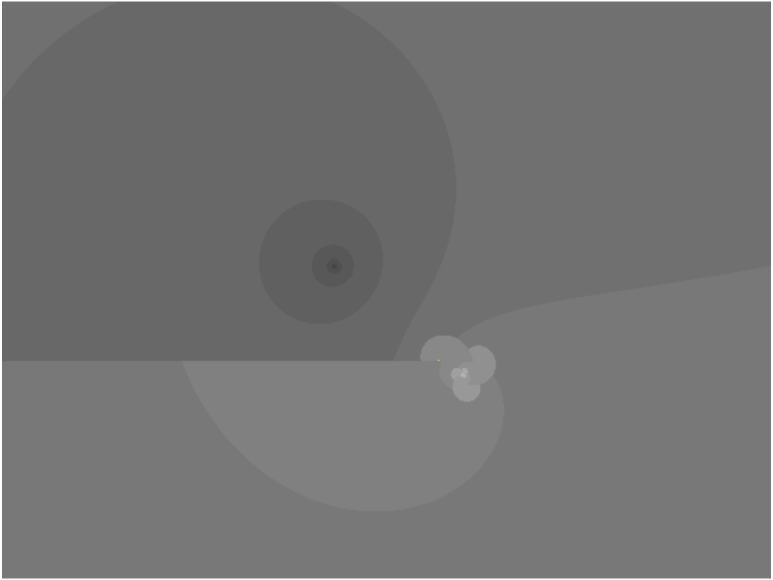
A tour of c values, $c = 1.87 + .10i$



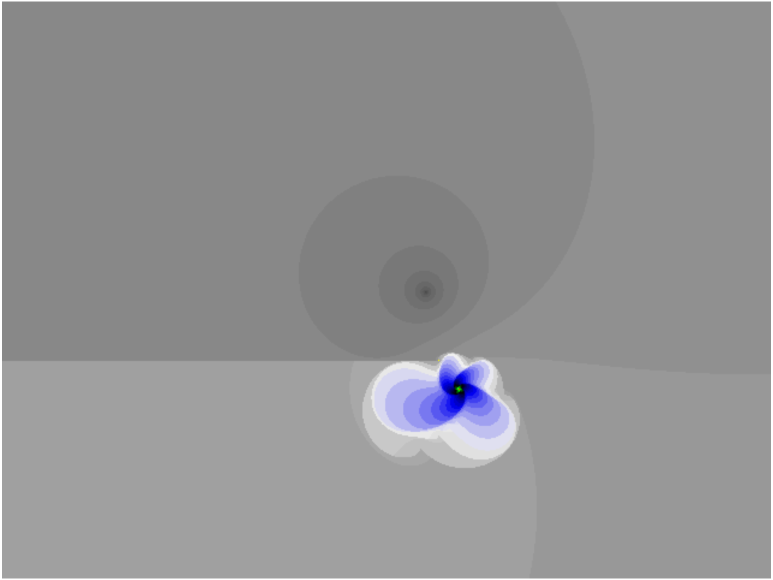
A tour of c values, $c = 1.96 - .11i$



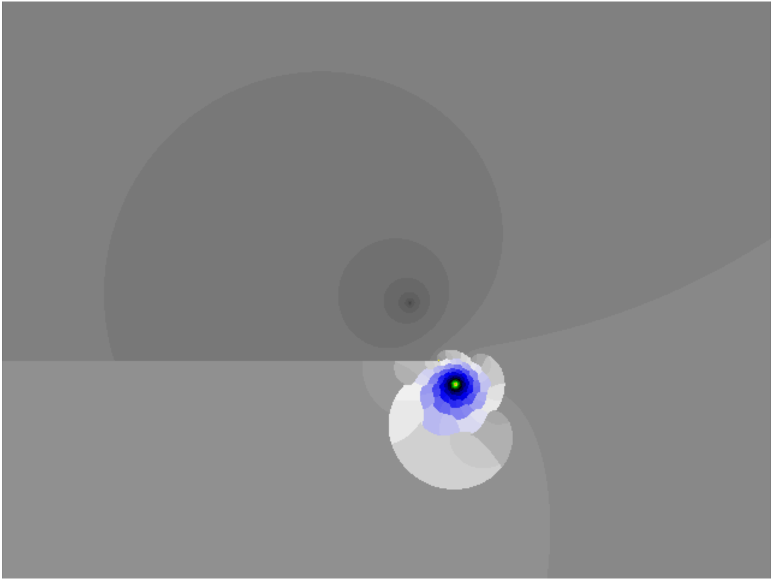
A tour of c values, $c = .29 + 1.45i$



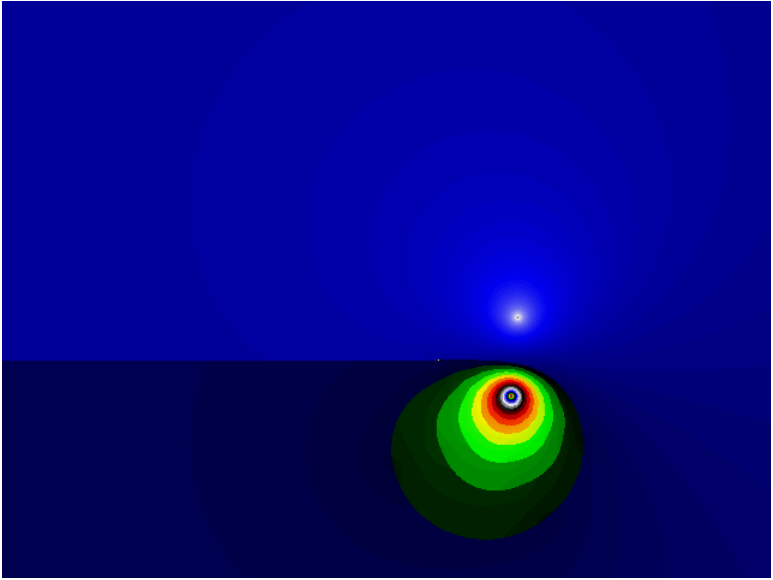
A tour of c values, $c = .90 + .58i$



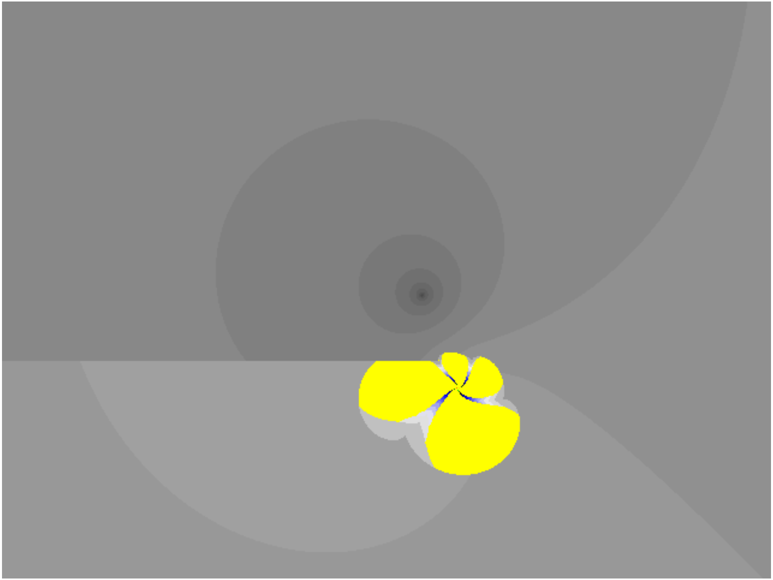
A tour of c values, $c = .64 + .62i$



A tour of c values, $c = 2.43 + .06i$



A tour of c values, $c = .84 + .58i$



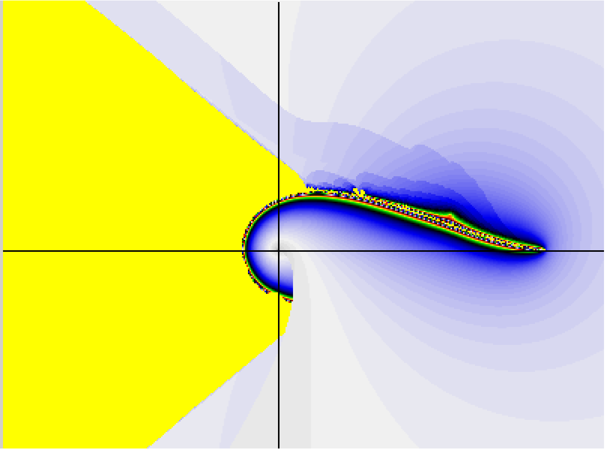
A tour of c values, $c = .02 + .74i$

A tour of c values, $c = -1.12 + .34i$

Index set

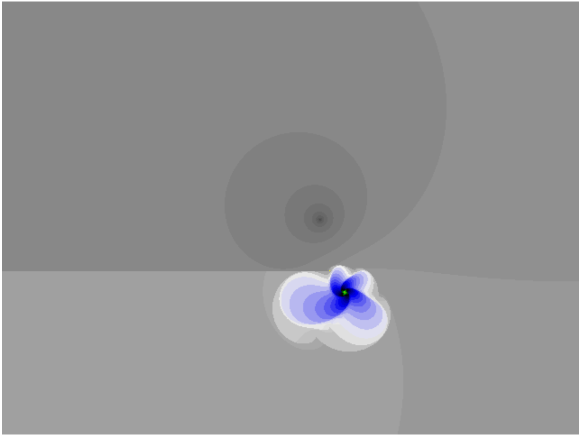
Want an idea of what to expect from a given c value.

Take $1.2 - 1.2i$, iterate it for each value of c, plot its color.

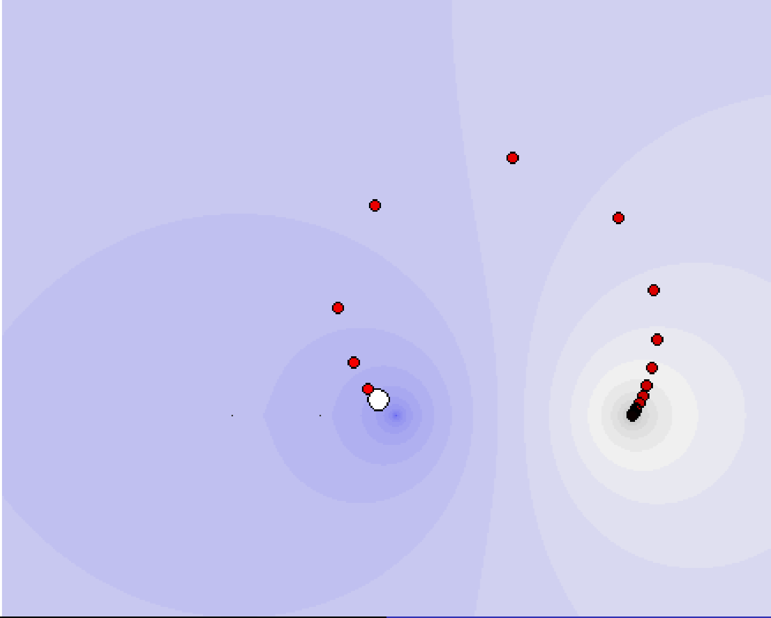


Source and a sink

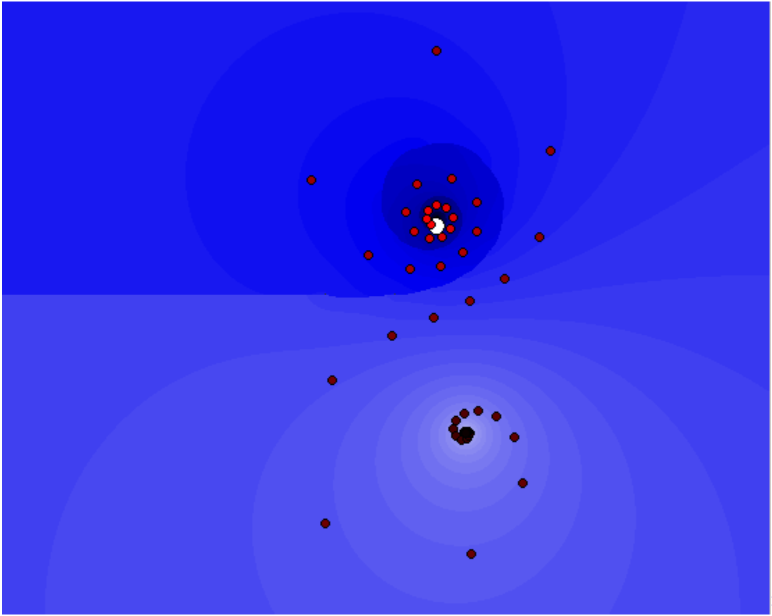
Most of the images have a source and a sink.



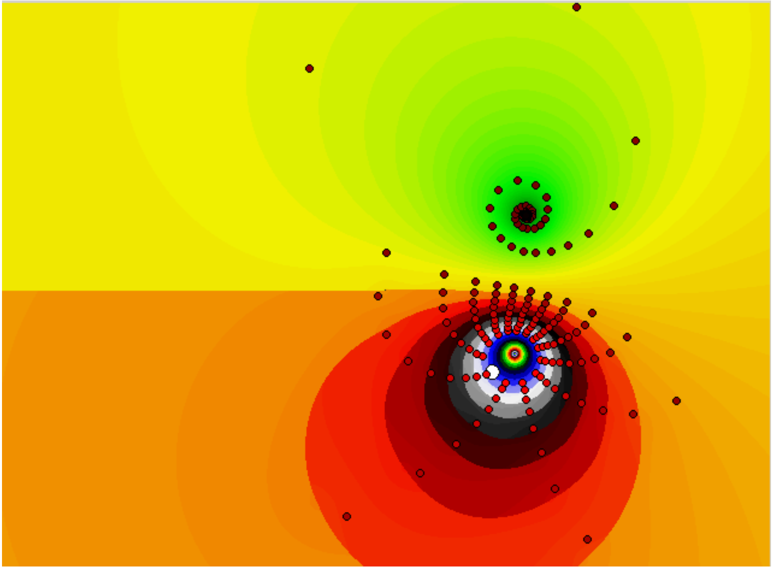
Source and a sink, $c = 3$



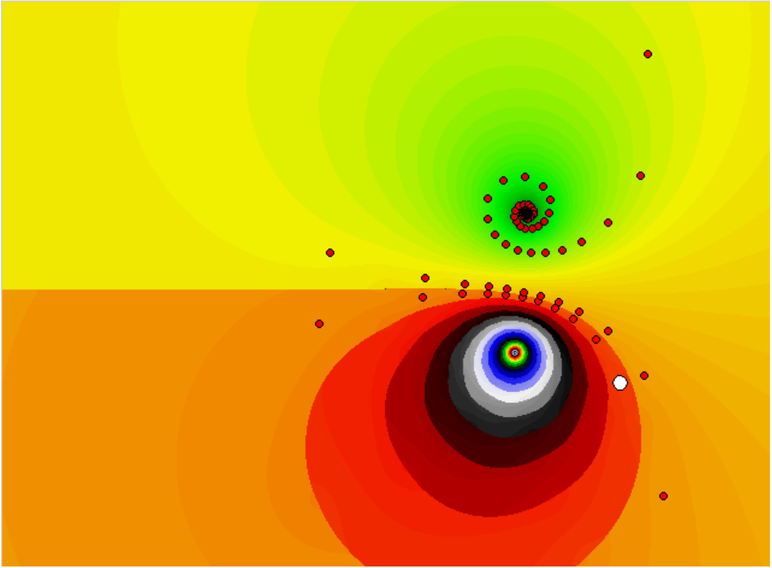
Source and a sink, $c = 2.16 - .3i$



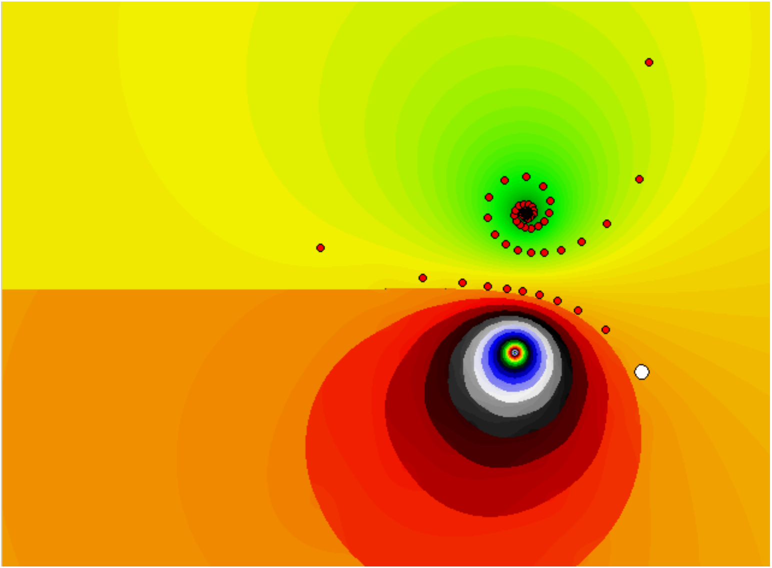
Source and a sink, $c = 2.43 + .06i$



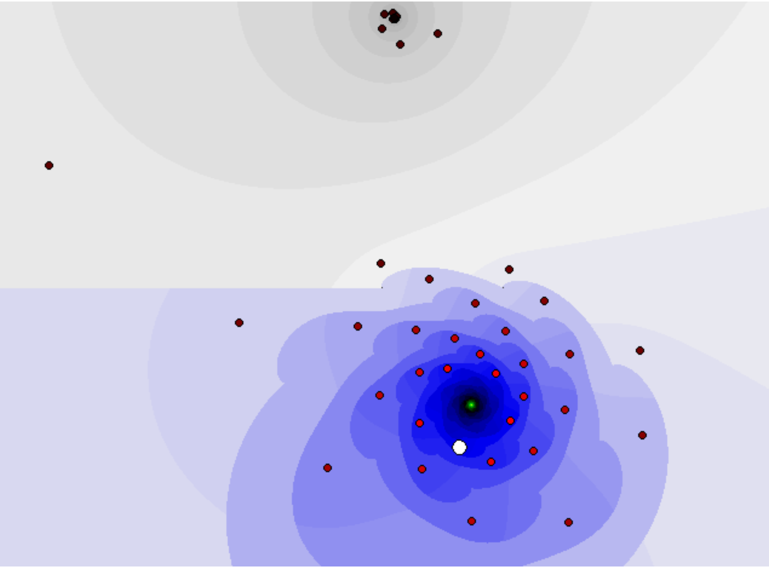
Source and a sink, $c = 2.43 + .06i$



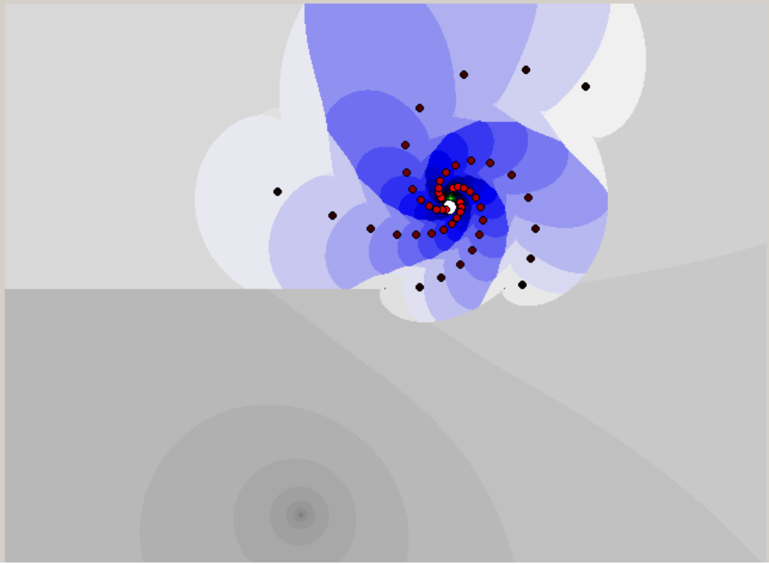
Source and a sink, $c = 2.43 + .06i$



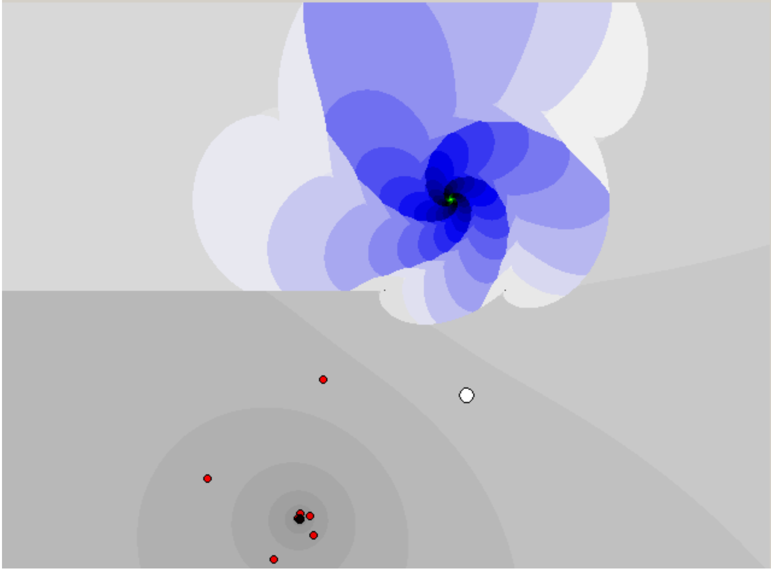
Source and a sink, $c = 1.17 + .55i$



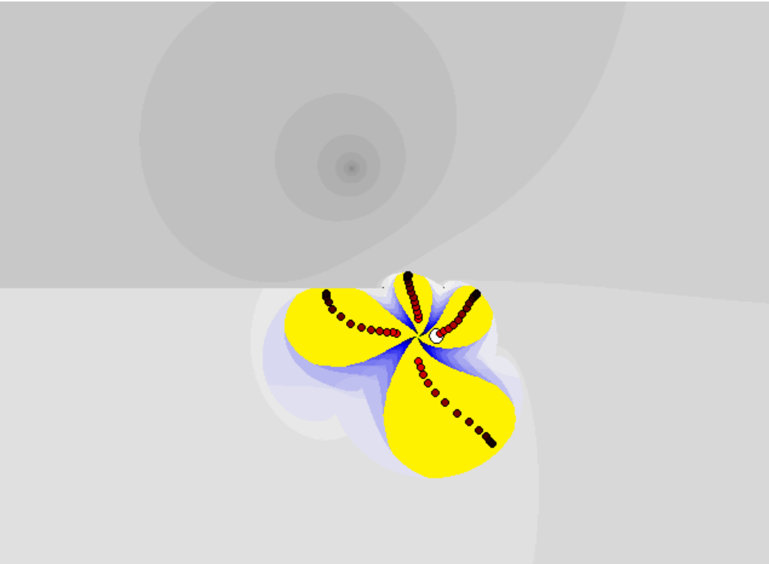
Source and a sink, $c = .75 - .64i$



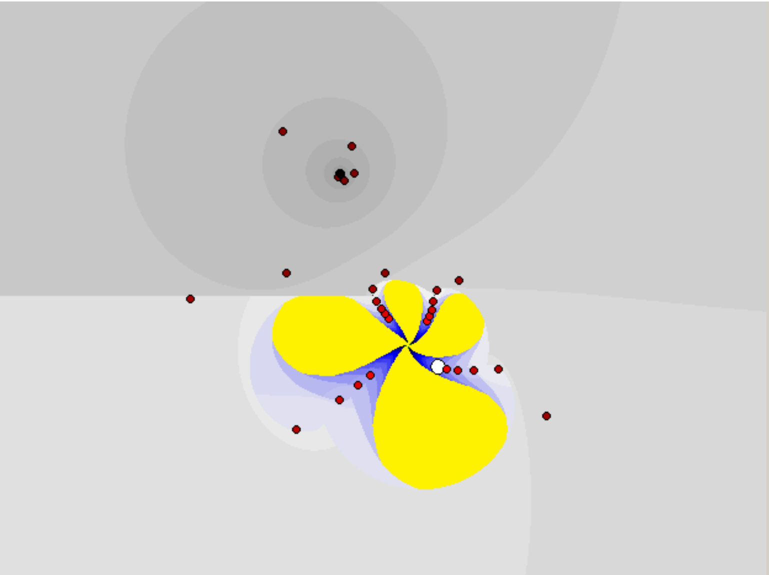
Source and a sink, $c = .75 - .64i$



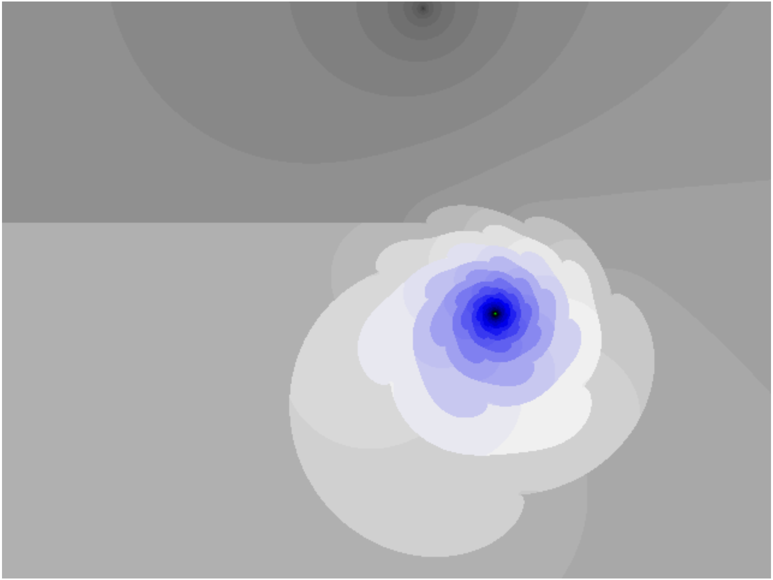
Source and a sink, $c = .84 + .61i$



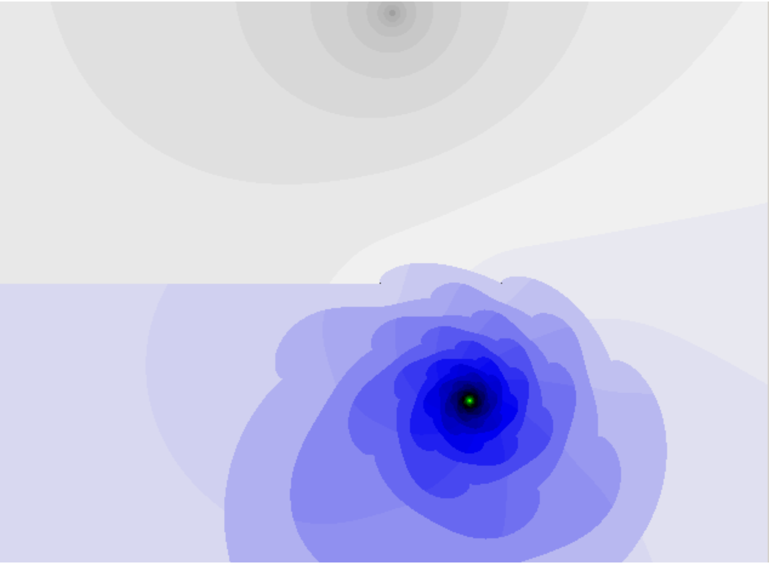
Source and a sink, $c = .84 + .61i$



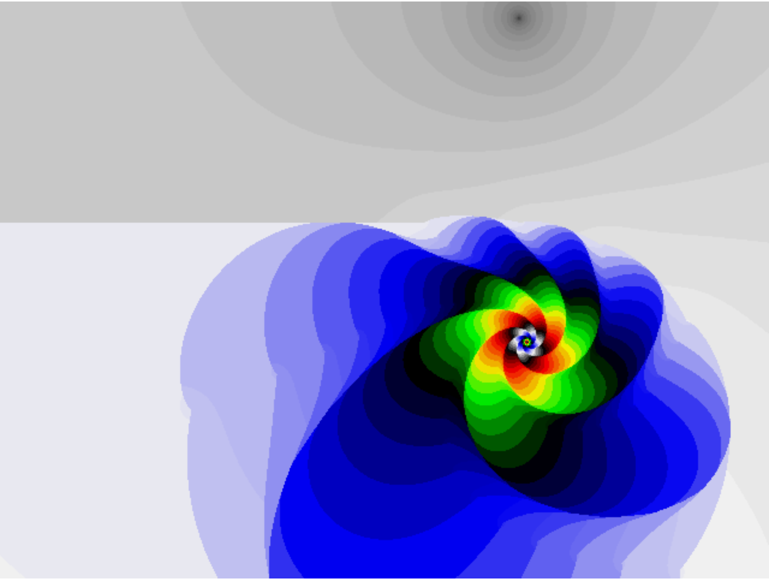
Source $c = 1.10 + .57i$



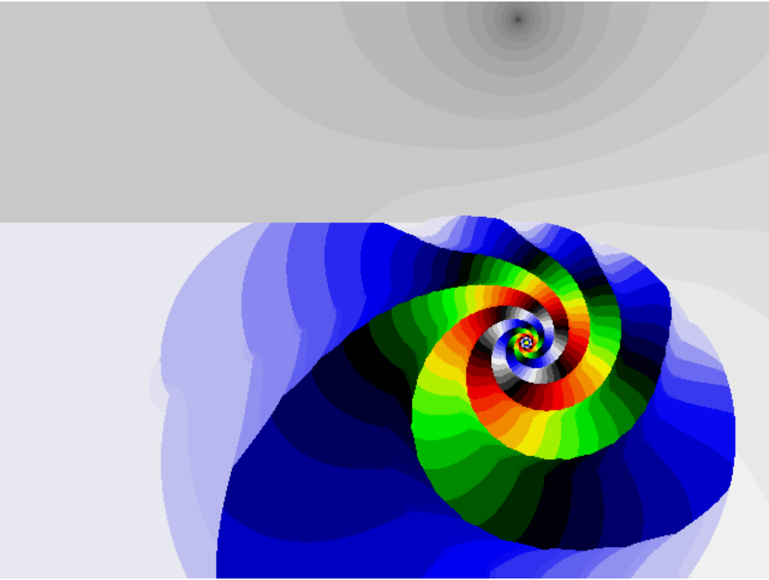
Source $c = 1.17 + .55i$



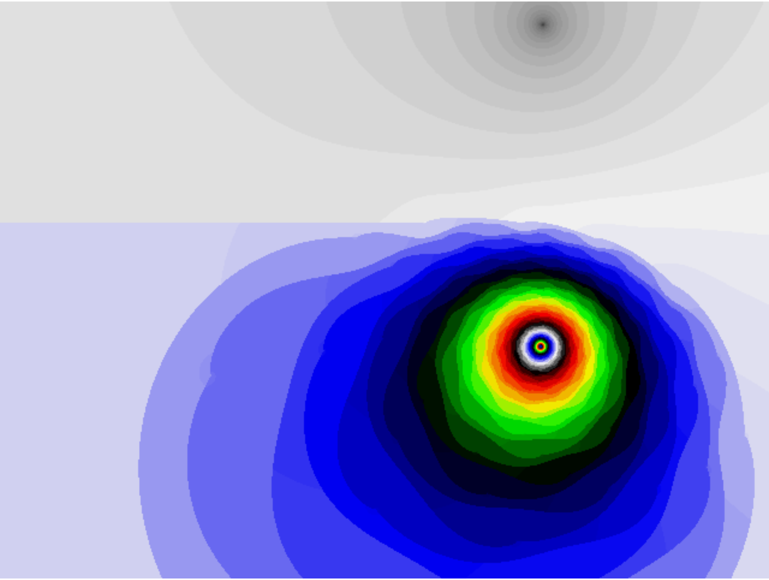
Source $c = 1.60 + .33i$



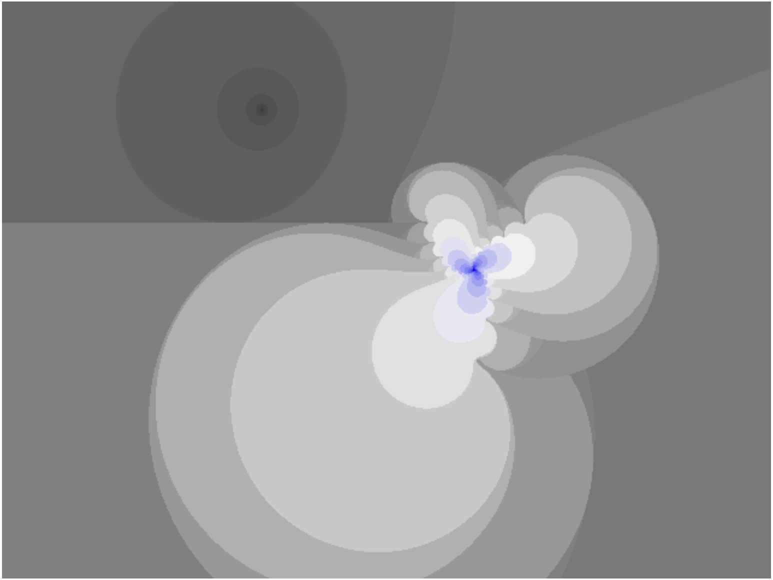
Source $c = 1.59 + .32i$



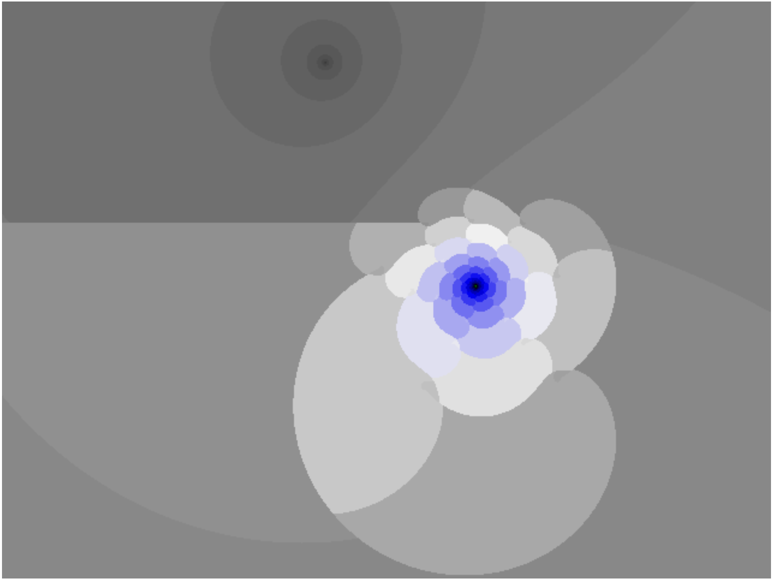
Source $c = 1.74 + .28i$



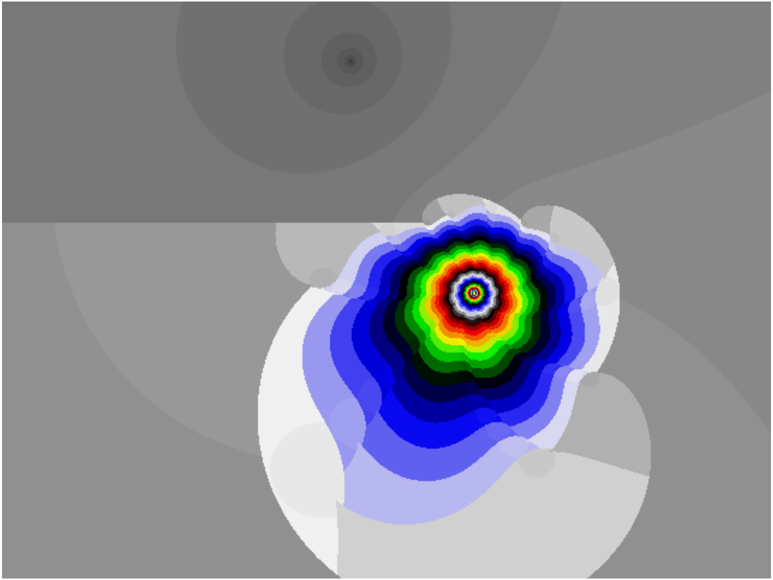
Source $c = .25 + .74i$



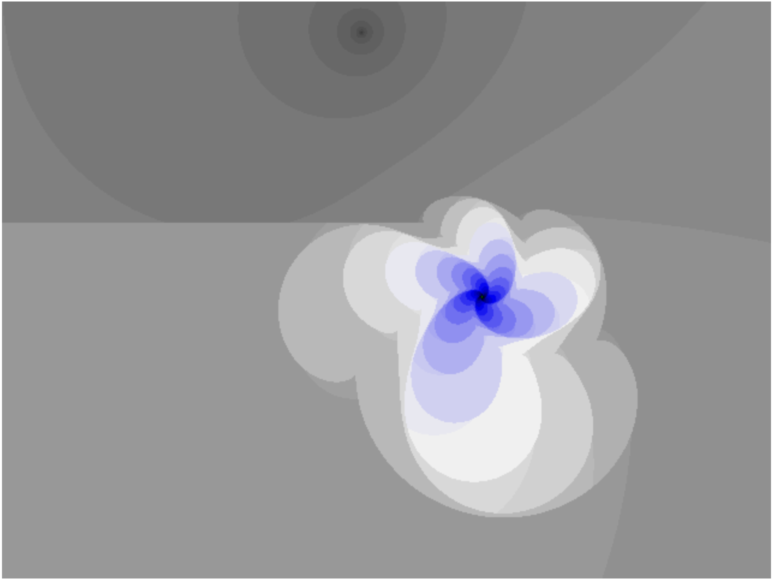
Source $c = .56 + .65i$



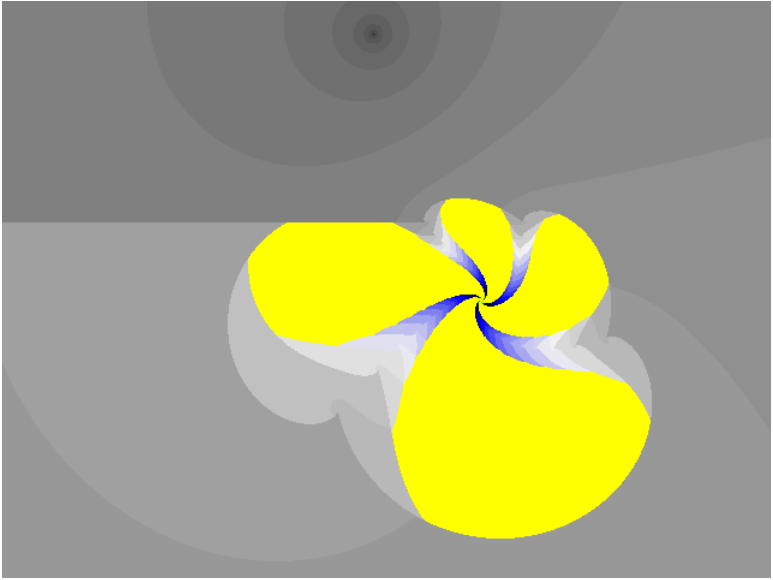
Source $c = .65 + .59i$



Source $c = .78 + .65i$



Source $c = .82 + .61i$

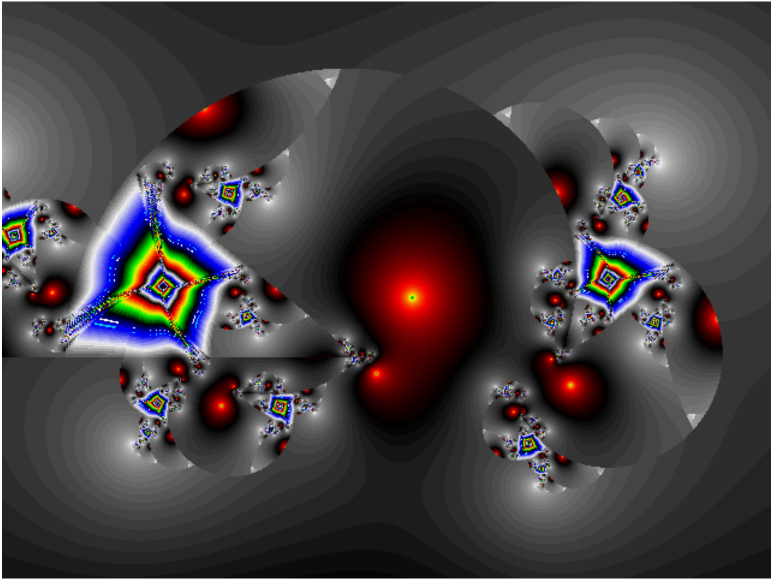


Complex sine

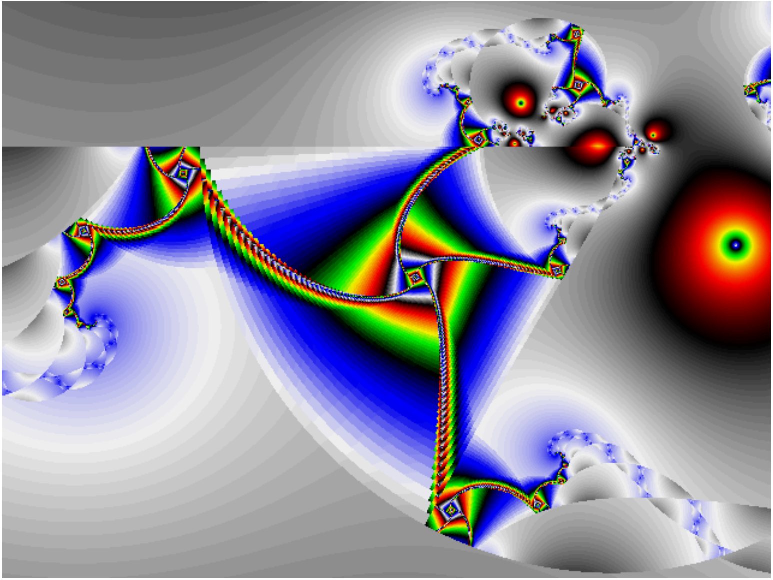
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

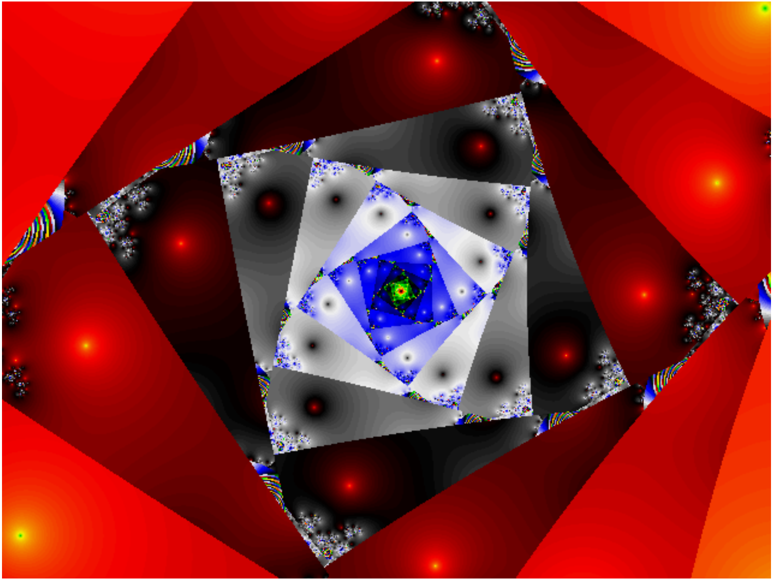
$c \sin(\ln z)$



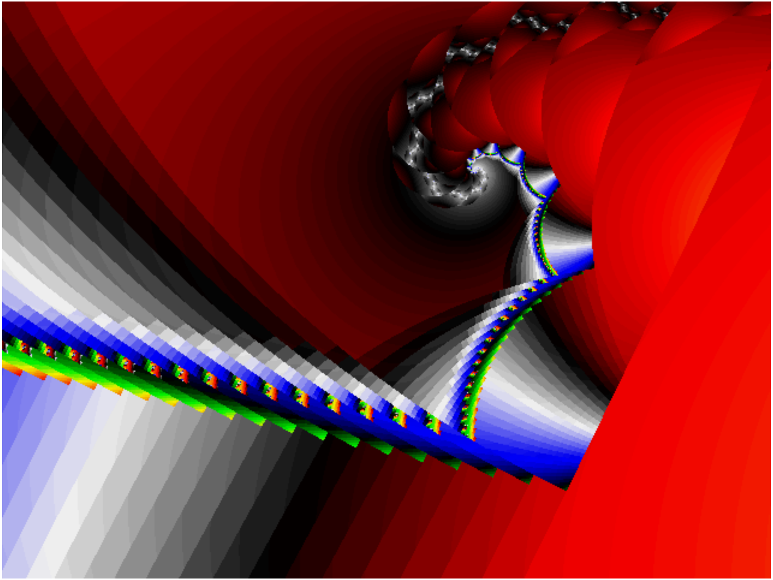
$c\sin(\ln z)$



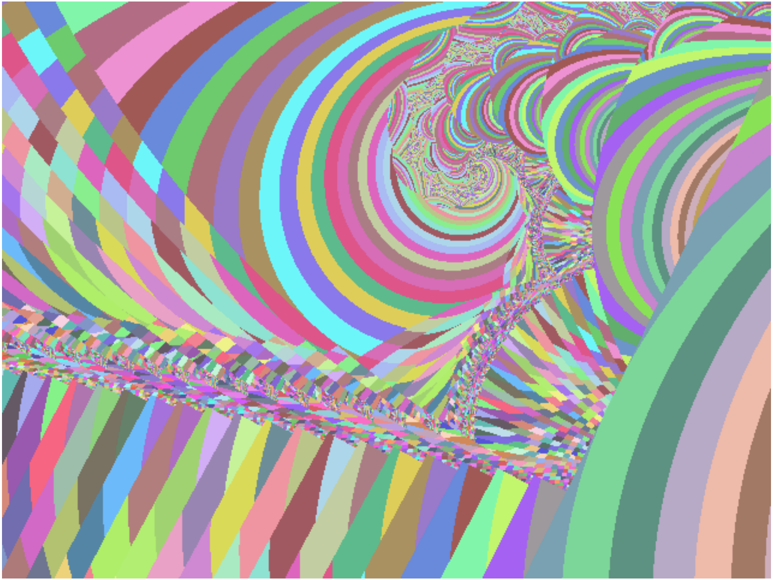
$c \sin(\ln z)$



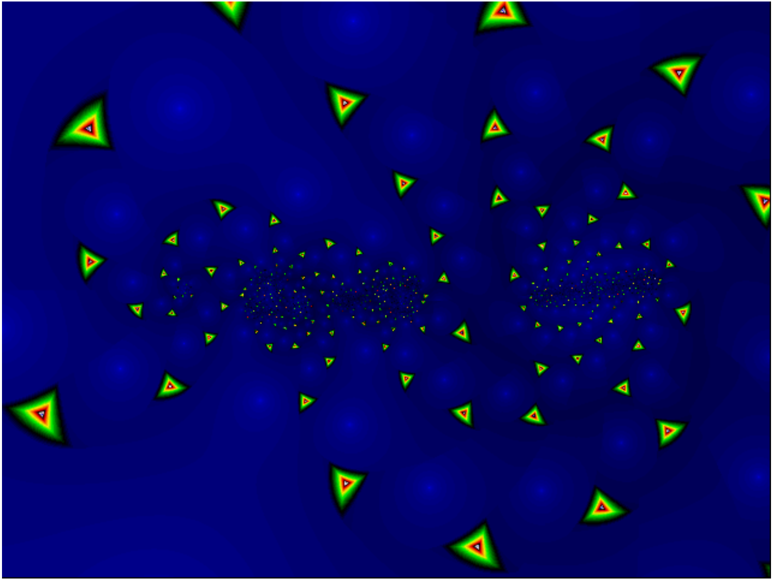
$c \sin(\ln z)$



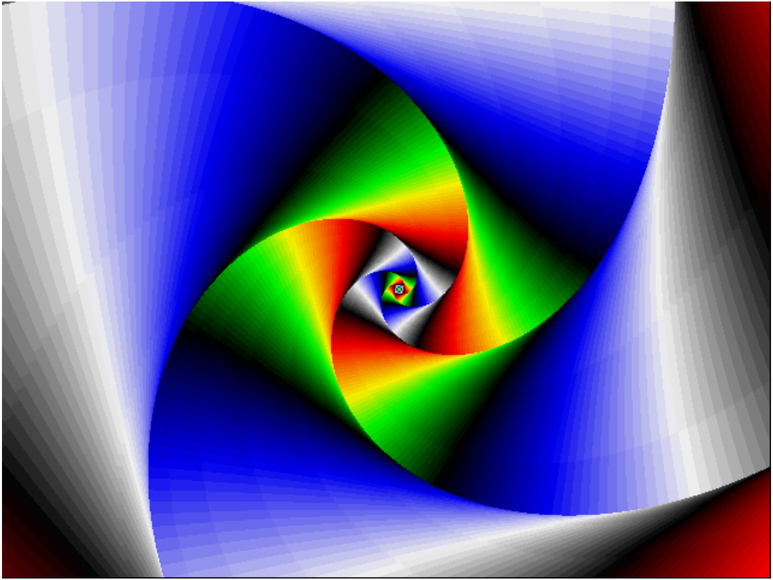
$c\sin(\ln z)$



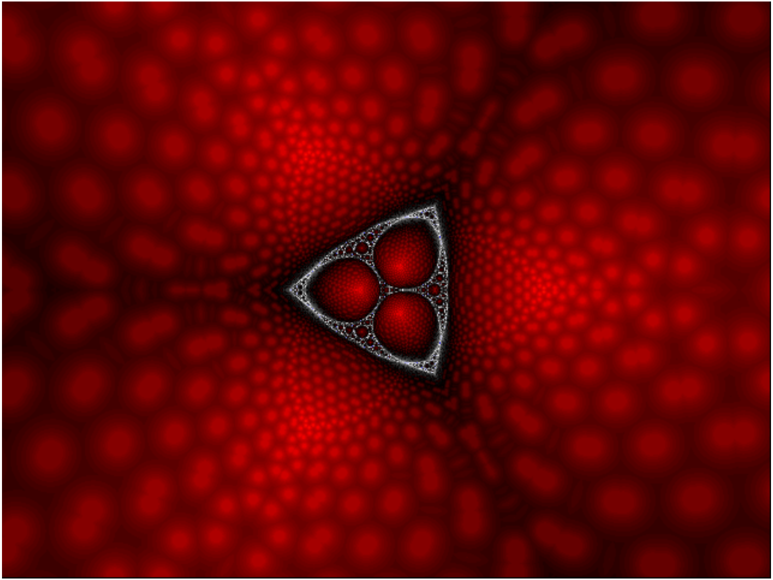
$c\sin(\ln(\sin(\ln z)))$



$\sin(\ln(\sin(\ln z)c))/c$



$c\ln(z^3)$



$c\ln(z^4)$