Probability Questions from the Game Pickomino

Brian Heinold

Department of Mathematics and Computer Science
Mount St. Mary's University

November 5, 2016

a.k.a. *Heckmeck am Bratwurmeck*
Created by Reiner Knizia
Published by *Zoch zum Spielen* and Rio Grande Games
2005

How to Play

- 8 dice, each with numbers 1-5 and a worm (worth 5)

- Tiles with costs 21-36, worth 1, 2, 3, or 4 points

How to Play, Continued

- Use the dice to acquire tiles.

How to Play, Continued

- Use the dice to acquire tiles.
- Roll the dice sort of Yahtzee-style, where you choose a group of same-sided dice.

How to Play, Continued

- Use the dice to acquire tiles.
- Roll the dice sort of Yahtzee-style, where you choose a group of same-sided dice.
- But once you take a side, you can't take that same side again in later rolls.

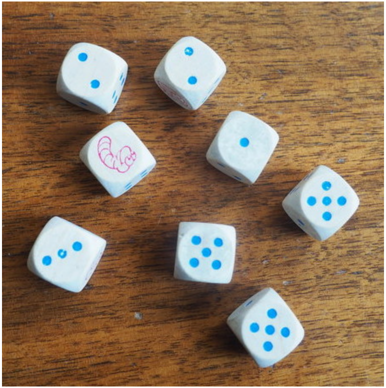
How to Play, Continued

- Use the dice to acquire tiles.
- Roll the dice sort of Yahtzee-style, where you choose a group of same-sided dice.
- But once you take a side, you can't take that same side again in later rolls.
- Reroll the dice not taken and repeat until (a) you have enough to take a tile or (b) you are unable to take any dice (because all the dice in a reroll match ones already taken)

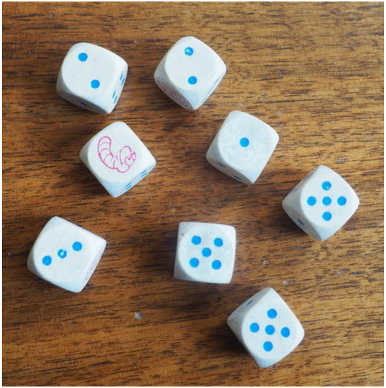
How to Play, Continued

- Use the dice to acquire tiles.
- Roll the dice sort of Yahtzee-style, where you choose a group of same-sided dice.
- But once you take a side, you can't take that same side again in later rolls.
- Reroll the dice not taken and repeat until (a) you have enough to take a tile or (b) you are unable to take any dice (because all the dice in a reroll match ones already taken)
- Key rule: You need at least one worm in order to take a tile.

Example Turn, First Roll

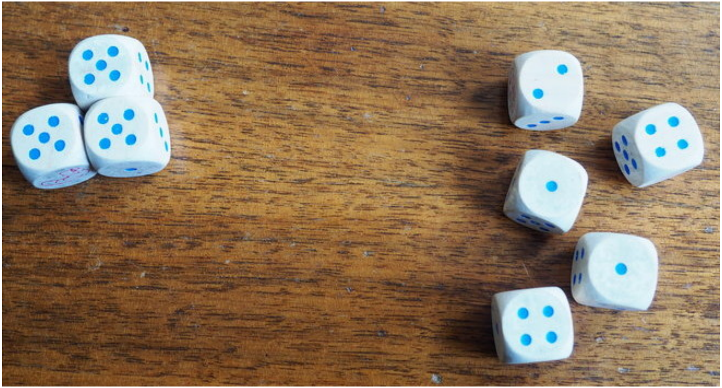


Example Turn, First Roll

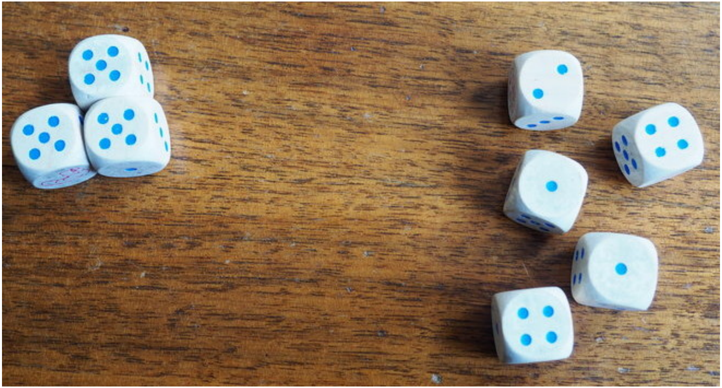


Take the three 5s.

Example Turn, Second Roll



Example Turn, Second Roll



Take the 4s. We're up to 23 points, but no worms, so we can't take anything.

Example Turn, Third Roll

Example Turn, Third Roll

Take the worm. We're up to 28 points. Let's grab the 28 tile. It's worth 2 points.

How to Play, Continued

- If you fail, you lose the tile you most recently acquired. It goes back to the center and the highest numbered tile is turned over (unless the tile returned was the highest).
- Fun note: if you get exactly the number of points as an opponent's most recently taken tile, you can steal it.
- Game ends when all tiles are taken. Winner is one with most points (total worms on tiles).

Basic Probability Questions

- Should we risk another roll?

- Should we risk another roll?
- We have two dice left and we want at least one 1, 2, or 3.

Basic Probability Questions

- Should we risk another roll?
- We have two dice left and we want at least one 1, 2, or 3.
- That is, we don't want both dice to be 4, 5, or a Worm.

Basic Probability Questions

- Should we risk another roll?
- We have two dice left and we want at least one 1, 2, or 3.
- That is, we don't want both dice to be 4, 5, or a Worm.
- Probability of that is $\left(\frac{3}{6}\right)^2 = \frac{1}{4}$.

Basic Probability Questions

- Should we risk another roll?
- We have two dice left and we want at least one 1, 2, or 3.
- That is, we don't want both dice to be 4, 5, or a Worm.
- Probability of that is $\left(\frac{3}{6}\right)^2 = \frac{1}{4}$.
- This is a nice example for any class covering basic probability.

Expected Value

- Let's consider taking one more roll:

Expected Value

- Let’s consider taking one more roll:
- If we succeed, we add 3 worms (24/36 probability) or 4 worms (3/36 probability) to our total.

Expected Value

- Let's consider taking one more roll:
- If we succeed, we add 3 worms (24/36 probability) or 4 worms (3/36 probability) to our total.
- If we fail (9/36 probability), we don't gain any points and we lose whatever is on the top of our stack (value n).

Expected Value

- Let's consider taking one more roll:
- If we succeed, we add 3 worms (24/36 probability) or 4 worms (3/36 probability) to our total.
- If we fail (9/36 probability), we don't gain any points and we lose whatever is on the top of our stack (value n).
- Expected value is $\frac{9}{36}(0-n)+\frac{24}{36}(3)+\frac{3}{36}(4)=2+\frac{1}{3}-\frac{n}{4}$.

Expected Value

- Let's consider taking one more roll:
- If we succeed, we add 3 worms (24/36 probability) or 4 worms (3/36 probability) to our total.
- If we fail (9/36 probability), we don't gain any points and we lose whatever is on the top of our stack (value n).
- Expected value is $\dfrac{9}{36}(0-n)+\dfrac{24}{36}(3)+\dfrac{3}{36}(4)=2+\dfrac{1}{3}-\dfrac{n}{4}$.
- So if we are holding no tiles or a 1-tile, it might make sense to go for more.

Expected Value

- Let’s consider taking one more roll:
- If we succeed, we add 3 worms (24/36 probability) or 4 worms (3/36 probability) to our total.
- If we fail (9/36 probability), we don’t gain any points and we lose whatever is on the top of our stack (value n).
- Expected value is $\frac{9}{36}(0-n)+\frac{24}{36}(3)+\frac{3}{36}(4)=2+\frac{1}{3}-\frac{n}{4}$.
- So if we are holding no tiles or a 1-tile, it might make sense to go for more. Losing that 1-tile might expose a high tile. . .

A Simple Question

- We need to get to 27.

A Simple Question

- We need to get to 27.
- The only way is two 2s.

A Simple Question

- We need to get to 27.
- The only way is two 2s.
- The probability is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$.

What Are the Chances *That* Would Happen?

- We had a 5 and two worms and our next roll was all 5s and worms!

What Are the Chances *That* Would Happen?

- We had a 5 and two worms and our next roll was all 5s and worms!
- What's the chance of that?

What Are the Chances *That* Would Happen?

- We had a 5 and two worms and our next roll was all 5s and worms!
- What's the chance of that?
- $\left(\frac{2}{6}\right)^5 = \frac{1}{243}$
- Seems like it happens to me more often than that!

This Will Never Happen

This Will Never Happen

- You will never see this unless you arrange the dice yourself.

This Will Never Happen

- You will never see this unless you arrange the dice yourself.
- $\left(\frac{1}{6}\right)^8$ = about 1 in 1.7 million

This Will Never Happen

- You will never see this unless you arrange the dice yourself.
- $\left(\frac{1}{6}\right)^8$ = about 1 in 1.7 million
- 6 or 7 worms in one roll are somewhat more likely:

$$\binom{8}{1}\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^1 \approx \frac{1}{42{,}000} \qquad \binom{8}{2}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^2 \approx \frac{1}{2{,}000}$$

This Will Never Happen

- You will never see this unless you arrange the dice yourself.
- $\left(\frac{1}{6}\right)^8$ = about 1 in 1.7 million
- 6 or 7 worms in one roll are somewhat more likely:

$$\binom{8}{1}\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^1 \approx \frac{1}{42{,}000} \qquad \binom{8}{2}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^2 \approx \frac{1}{2{,}000}$$

- These are nice examples of binomial probabilities.

A Slightly Trickier Basic Probability Question

- In this situation we need a worm and only have 2 dice left.

- In this situation we need a worm and only have 2 dice left.
- Probability of a worm on the first roll is $1-\left(\frac{5}{6}\right)^6 \approx 30\%$.

A Slightly Trickier Basic Probability Question

- In this situation we need a worm and only have 2 dice left.
- Probability of a worm on the first roll is $1 - \left(\frac{5}{6}\right)^6 \approx 30\%$.
- If we roll all 4s and 5s, we're sunk. Probability: $\left(\frac{2}{6}\right)^2 \approx 10\%$.

A Slightly Trickier Basic Probability Question

- In this situation we need a worm and only have 2 dice left.
- Probability of a worm on the first roll is $1 - \left(\frac{5}{6}\right)^6 \approx 30\%$.
- If we roll all 4s and 5s, we're sunk. Probability: $\left(\frac{2}{6}\right)^2 \approx 10\%$.
- The other 60% of the time, we get another reroll, with a 1/6 probability, so this adds an additional 10%.

A Slightly Trickier Basic Probability Question

- In this situation we need a worm and only have 2 dice left.
- Probability of a worm on the first roll is $1 - \left(\frac{5}{6}\right)^6 \approx 30\%$.
- If we roll all 4s and 5s, we're sunk. Probability: $\left(\frac{2}{6}\right)^2 \approx 10\%$.
- The other 60% of the time, we get another reroll, with a $1/6$ probability, so this adds an additional 10%.
- So, in total, about a 40% chance.

Simulation (because I don't trust my calculations)

```python
from random import randint
count = 0
n = 100000
for i in range(n):
    d1 = randint(1,6)
    d2 = randint(1,6)

    if d1 in [4,5] and d2 in [4,5]:
        continue

    if d1 == 6 or d2 == 6:
        count += 1
        continue

    if randint(1,6) == 6:
        count += 1

print (count/n * 100)
```

When Does It Make Sense to Take Something Small?

- In the roll above it might make sense to take the 1 instead of the two 5s.

When Does It Make Sense to Take Something Small?

- In the roll above it might make sense to take the 1 instead of the two 5s.
- Why? It gives you a better chance of getting a worm.

When Does It Make Sense to Take Something Small?

- In the roll above it might make sense to take the 1 instead of the two 5s.
- Why? It gives you a better chance of getting a worm.
- Answer, though, would depend on what tiles are available.

When Does It Make Sense to Take Something Small?

- An interesting project would be to identify when it would make sense to take a single 1 or 2 instead of bigger things.

When Does It Make Sense to Take Something Small?

- An interesting project would be to identify when it would make sense to take a single 1 or 2 instead of bigger things.
- Above on left, if you need a lot of points, the 1 might be the best option.

When Does It Make Sense to Take Something Small?

- An interesting project would be to identify when it would make sense to take a single 1 or 2 instead of bigger things.
- Above on left, if you need a lot of points, the 1 might be the best option.
- On the right, taking a 5 would seriously limit your future points.

Questions for the First Roll

- Take the two worms or the three 5s?

Questions for the First Roll

- Take the two worms or the three 5s?
- In general, n worms versus $n + k$ 5s?

Questions for the First Roll

- Take the two worms or the three 5s?
- In general, n worms versus $n + k$ 5s?
- Similarly, two 5s versus three 4s?

Questions for the First Roll

- Take the two worms or the three 5s?
- In general, n worms versus $n+k$ 5s?
- Similarly, two 5s versus three 4s?
- Three 4s versus 1 worm?

Questions for the First Roll

- Take the two worms or the three 5s?
- In general, n worms versus $n+k$ 5s?
- Similarly, two 5s versus three 4s?
- Three 4s versus 1 worm?
- This would make another nice project.

More Questions for the First Roll

- Is it worth taking all those 5s? It seriously limits your chances of getting a worm?

More Questions for the First Roll

- Is it worth taking all those 5s? It seriously limits your chances of getting a worm?
- What about four 4s to start?

- I once played a two-player game, where between the two of us, we took all but one tile (and only lost that on the second to last turn). We only failed to take a tile once in all the turn.

A Two-Player Game Question

- I once played a two-player game, where between the two of us, we took all but one tile (and only lost that on the second to last turn). We only failed to take a tile once in all the turn.
- How unusual was that?

A Two-Player Game Question

- I once played a two-player game, where between the two of us, we took all but one tile (and only lost that on the second to last turn). We only failed to take a tile once in all the turn.
- How unusual was that?
- Tricky to answer because of the possibility of stealing tiles.

More Questions

- What type of final scores are typical in a two-player game? The highest I've seen is in the low 20s.

More Questions

- What type of final scores are typical in a two-player game? The highest I've seen is in the low 20s.
- What is the probability that someone can take the 36 tile?

More Questions

- What type of final scores are typical in a two-player game? The highest I've seen is in the low 20s.
- What is the probability that someone can take the 36 tile?
- What is the probability that you at least get up to 21? Or at least x?

Things People Have Done

- Cardon, Chetcuti-Sperandio, Delorme, and Lagrue have a few articles on modeling Pickomino as a Markov Chain.

- See, e.g., *A Markovian Process Modeling for Pickomino*, Proceedings of the 7th International Conference on Computers and Games, in Lecture Notes in Computer Science, pp. 199-210, Springer-Verlag, 2010.

  > *We provide in this paper original solutions to both problems: we provide (1) a compact representation of states and (2) a constructive method to compute the probability distributions, based on the partitioning of the space of roll results depending on a set of marked values. Finally, we show the efficiency of the proposed method through numerous experimental results: it turns out to be impressive compared to previous programs we developed.*

Another article by the same authors: *Determination and evaluation of efficient strategies for a stop or roll dice game: Heckmeck am Bratwurmeck (Pickomino)*, Computational Intelligence and Games, 2008. CIG '08. IEEE Symposium On Computational Intelligence and Games.

> *We propose then an algorithm using a Monte-Carlo method to evaluate probabilities of dice rolls and the accessibility of resources. By using this tactical computing in different ways the programs can play according to the stage of the game (beginning or end). Finally, we present experimental results comparing all the proposed algorithms. Over 7,500,000 matches opposed the different AIs and the winner of this contest turns out to be a strong opponent for human players.*

Things People Have Done

Thomas ten Cate at the blog Frozen Fractal (`http://frozenfractal.com/blog/2015/5/3/how-to-win-at-pickomino/`) created a computer player, using a dynamic programming approach to traverse the game tree. It seems to be reasonably effective. His code is on GitHub.

In Summary, Some Things You Can Do With Pickomino

- Play the game in class: There are lots of interesting easy-to-answer questions that come up.

In Summary, Some Things You Can Do With Pickomino

- Play the game in class: There are lots of interesting easy-to-answer questions that come up.
- There are various questions that would make good projects.

In Summary, Some Things You Can Do With Pickomino

- Play the game in class: There are lots of interesting easy-to-answer questions that come up.
- There are various questions that would make good projects.
- It would be interesting to program a computer to play the game, and compare various strategies (e.g. Markov vs. searching probability tree).

Thanks!