Patterns and Number Theory

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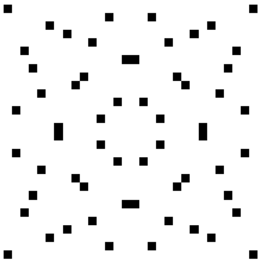
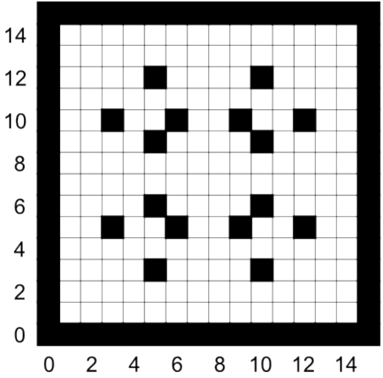


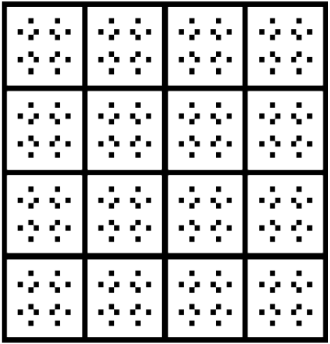
This is work with Jackie Kearney who researched this for her senior honors project.

Definition

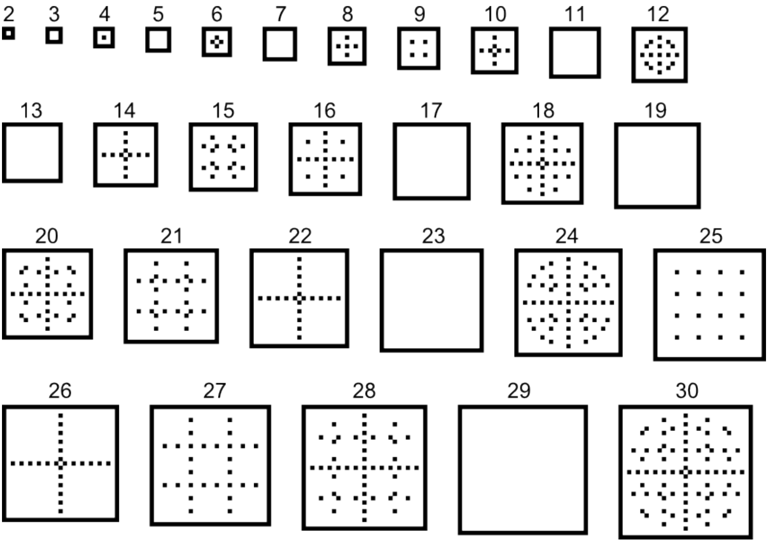
Given a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and an integer n, a *divisibility plot* is a plot of all the points (x, y) for which $n \mid f(x, y)$.

Divisibility plot of $f(x, y) = xy$ for $n = 15$

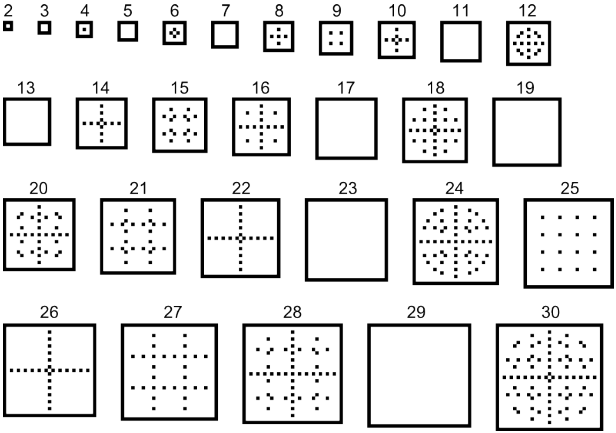




Divisibility plots of $f(x, y) = xy$ for n from 2 to 30



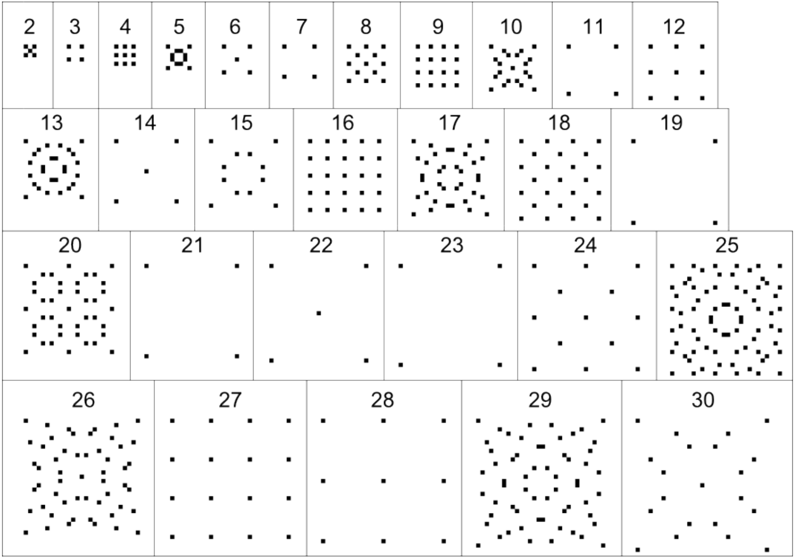
Divisibility plots of $f(x, y) = xy$ for n from 2 to 30



Consequences of Euclid's lemma:

- Primes are blank
- Grid pattern for perfect squares

$f(x, y) = x^2 + y^2$ for n from 2 to 30



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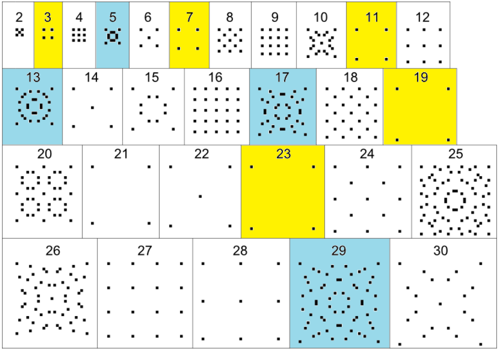
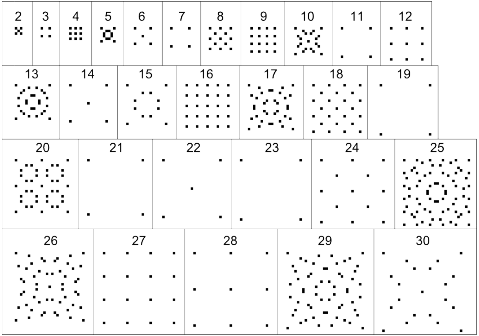


$4k+3$ primes – mostly empty
$4k+1$ primes – lots of points
Theorem of Fermat: Odd primes are sum of two squares iff they are of the form $4k+1$

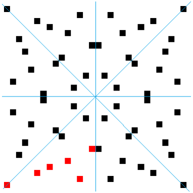
$f(x, y) = x^2 + y^2$ for n from 2 to 30



Generalization of Fermat's theorem: An integer n can be written as the sum of two squares if and only if each $4k + 3$ prime in the prime factorization of n is raised to an even power.

Explains why 21 is blank, explains various grid patterns.

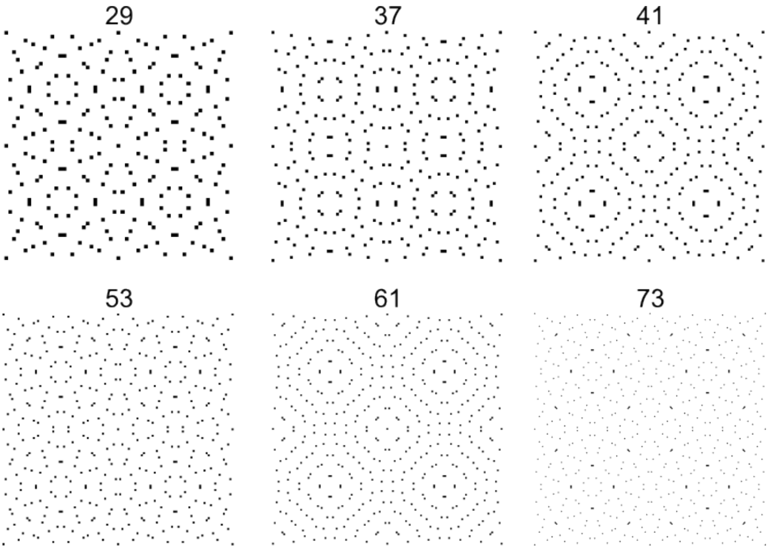
Only a few points matter



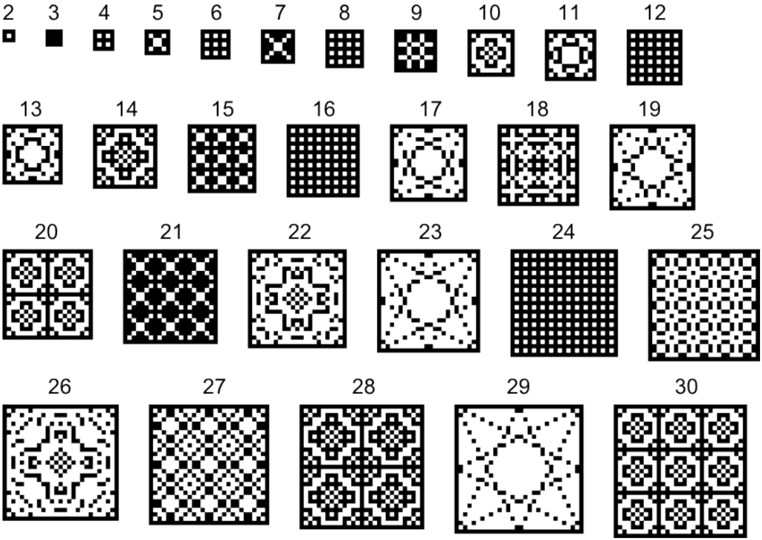
Each plot actually boils down to a handful of points and the symmetry arises from various transformations of those points.

- Symmetry about the lines $x = (p-1)/2$, $y = (p-1)/2$, and $y = \pm x$.
- If (x_0, y_0) satisfies $n \mid (x^2 + y^2)$, then (ax_0, ay_0) does, too.
- For each x value we get exactly one y in the range 0 to $(p-1)/2$ such that $p \mid (x^2 + y^2)$ (can show using Euler's criterion for quadratic residues, Fermat's Little Theorem, and Lagrange's theorem)

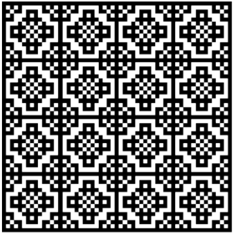
$4k + 1$ primes (plot range is 0 to $2n$)



$f(x,y) = xy(x^2 - 4y^2)(4x^2 - y^2)$ for n from 2 to 30



$f(x, y) = xy(x^2 - 4y^2)(4x^2 - y^2)$ for a few values of n



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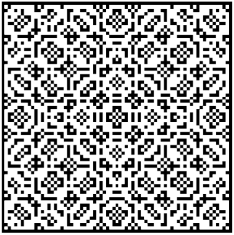

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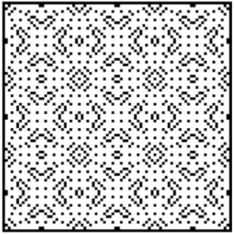

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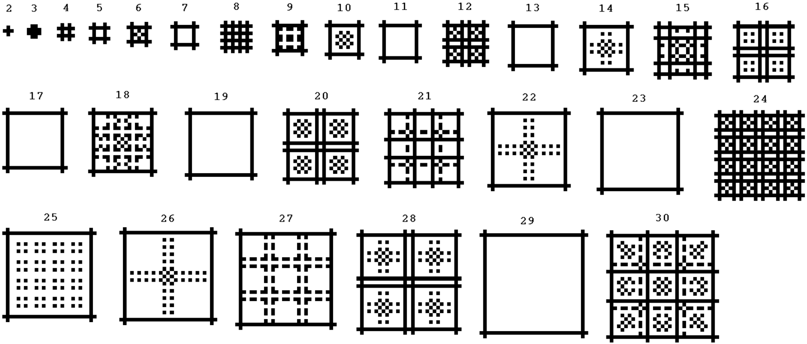
$f(x, y) = (x^2 - 1)(y^2 - 1)$ for n from 2 to 30



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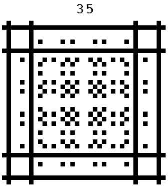
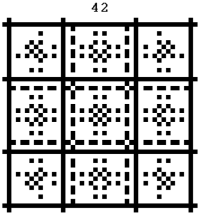
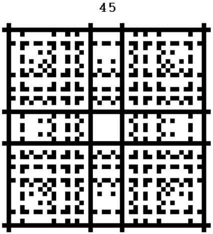
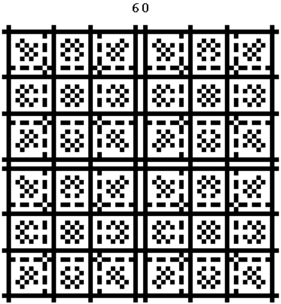
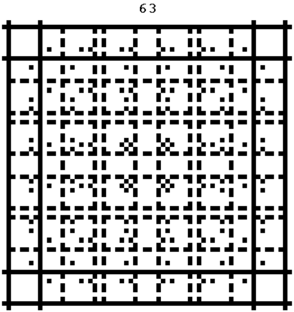


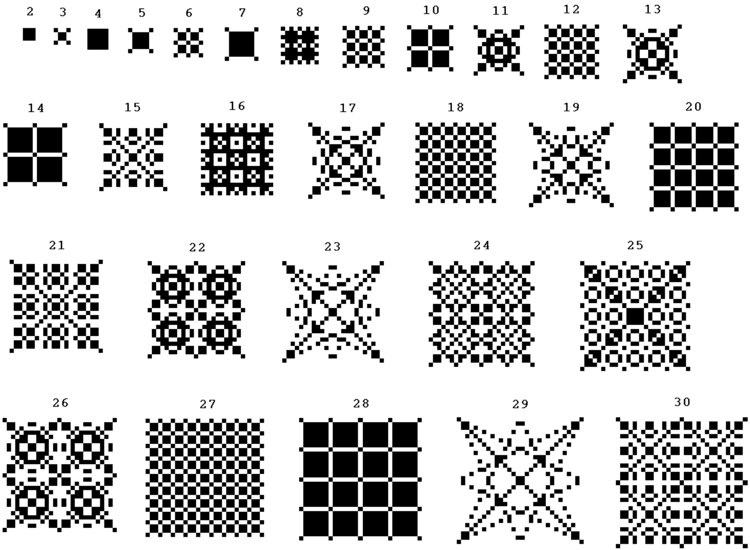




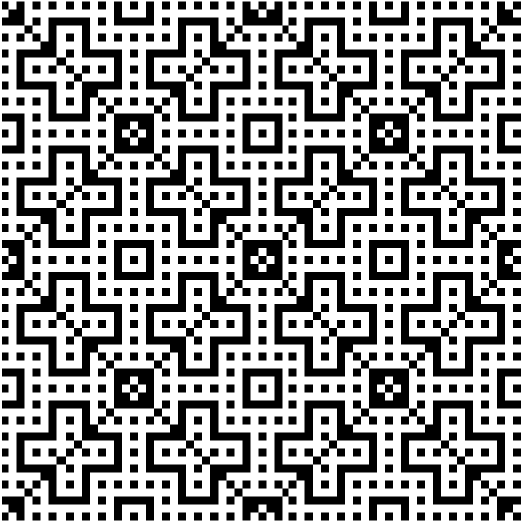




$f(x,y) = (x^2 - y^2)(x^2 - 4y^2)(4x^2 - y^2)$ for n from 2 to 30



$f(x,y) = (x^2 - y^2)(x^2 - 4y^2)(4x^2 - y^2)$ for $n = 64$



$f(x, y) = xy(4x^4 + 2xy + 4y^4)$ for $n = 32$