Surprises from iterating discontinuous functions

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Complex numbers

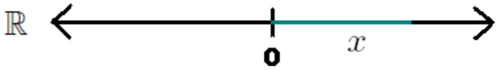
$i = \sqrt{-1}$ (solution to $x^2 + 1 = 0$)

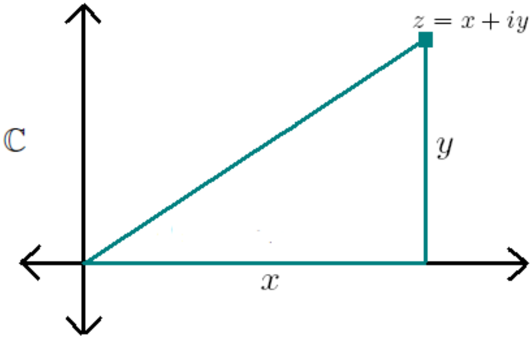
Examples: $2i, \quad 3 + 4i, \quad -.2 + .76i$

Addition: $(2 + 3i) + (5 + 8i) = 7 + 11i$

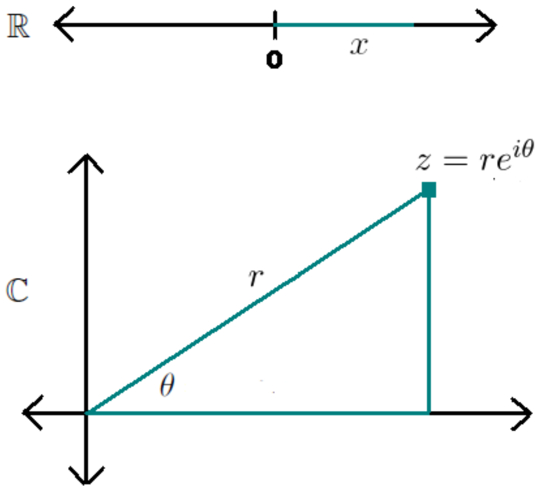
Multiplication: $(2 + 3i)(5 + 8i) = 10 + 31i + 24i^2 = -14 + 31i$

Picturing them





Picturing them



Iteration

Example: Let $f(x) = x^2$ and start with $x = 2$.

$f(2) = 4$
$f(4) = 16$
$f(16) = 256$
$f(256) = 65536$
$\ldots$

Iterates are approaching ∞.

A different starting point

Let $f(x) = x^2$ and start with $x = \frac{1}{2}$.

$f(\frac{1}{2}) = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{16}$

$f(\frac{1}{16}) = \frac{1}{256}$

$f(\frac{1}{256}) = \frac{1}{65536}$

$\ldots$

Iterates are approaching 0.

Another example

Let $f(x) = -x$ and start with $x = 1$.

$f(1) = -1$
$f(-1) = 1$
$f(1) = -1$
$f(-1) = 1$
$\ldots$

Iterates are not settling down on a value.

Coloring by convergence

Color each point according to how fast it converges.

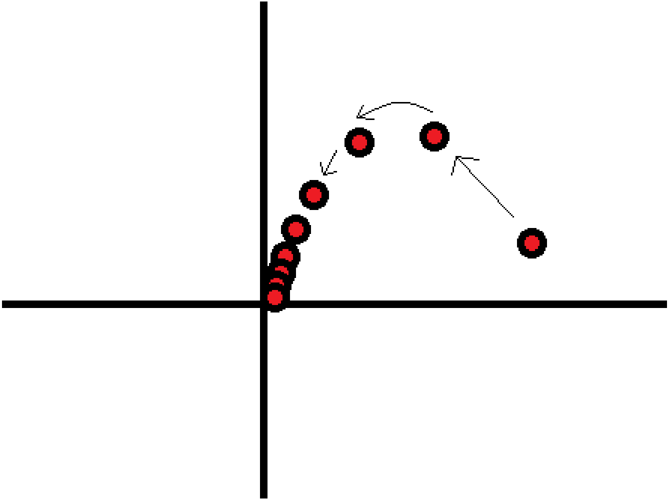
Count how many iterations until two successive values are within .00001 of each other.

Assign each count a color.

Convergence to infinity is still convergence (color by # of steps to exceed $\pm 10^5$).

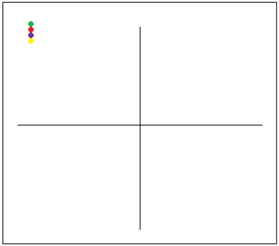
Iteration with complex numbers

Plug $z = x + iy$ into $f(z)$. Get a value, and plug that value into the function. Then plug the result of that into the function, etc.



The process

Look at all the possible starting values in a region.

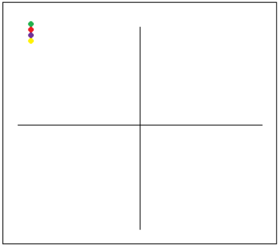


For each starting point, iterate the function.

If two successive values are within .00001 of each other, there's a very good chance that the iterates will converge.

The process, continued

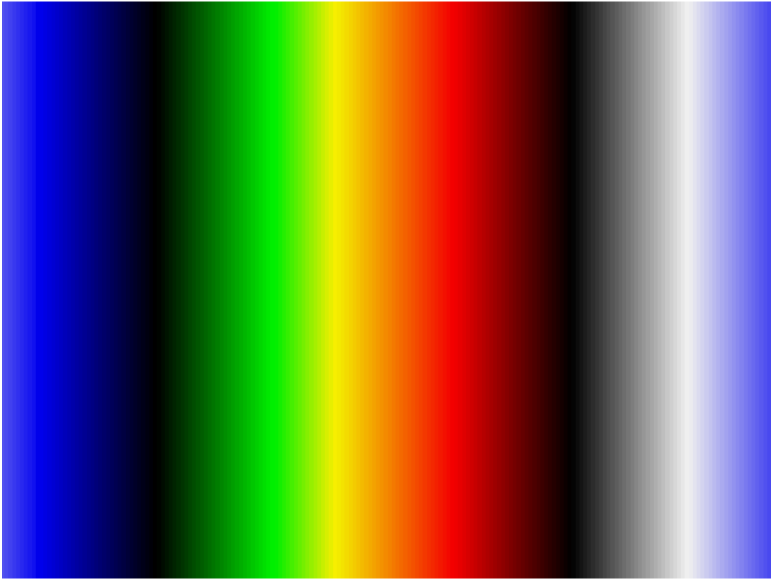
In this case, color the point with a color representing how long it took for this to happen.



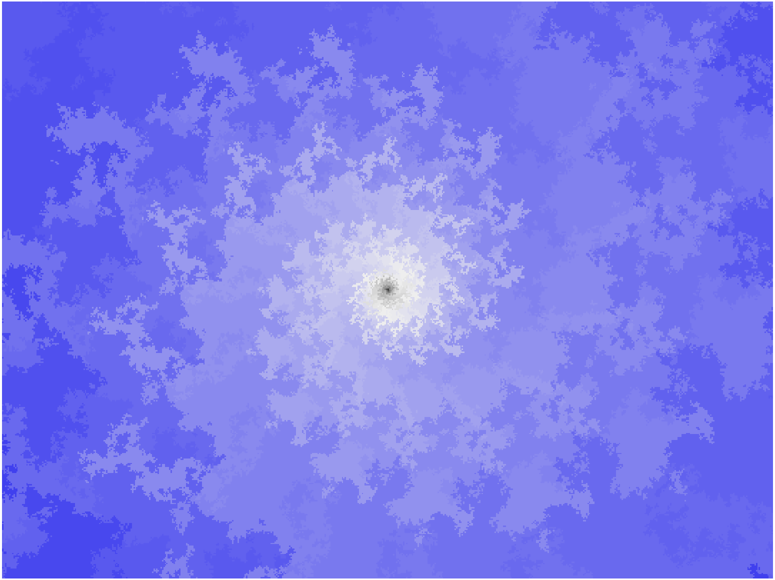
It is possible that the iteration may never stop. Give up after a few hundred iterations and color the point yellow.

Note: convergence to infinity is still convergence (color by how many steps for iteration to exceed $\pm 10^5$).

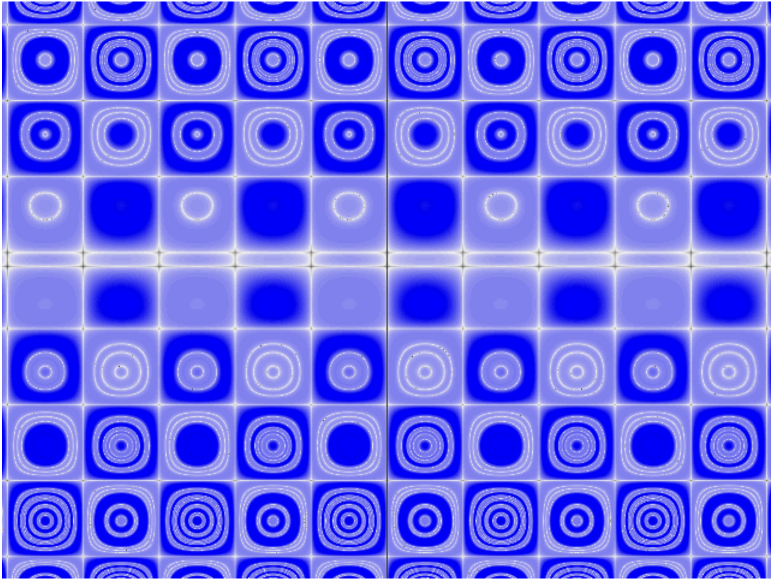
Color scheme



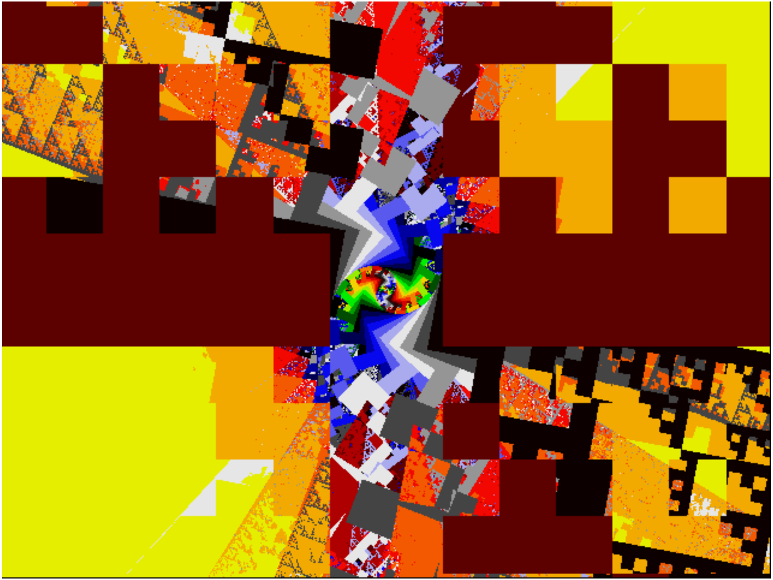
$f(x+iy) = c(\lfloor x \rfloor + \lfloor y \rfloor)$, $c = .77 + .35i$



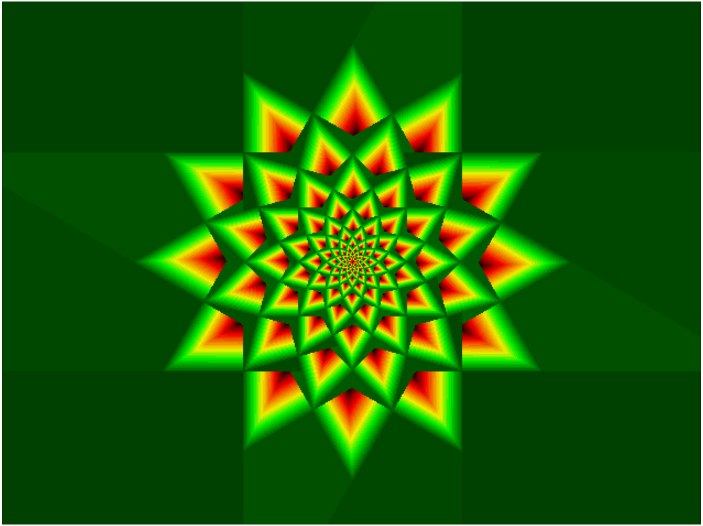
$f(x+iy) = c(\sin x)(\cos y)(1-y),\ c = .76 - .53i$



$f(x+iy) = (x+iy)(\chi_{(-1,1)}(x) + (x\&y)),\ c = .76 - .53i$

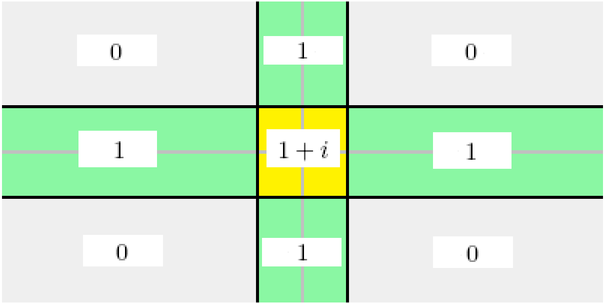


$f(x+iy) = cz(\chi_{(-1,1)}(x) + i\chi_{(-1,1)}(y)),\ c = .870 + .504i$



The function

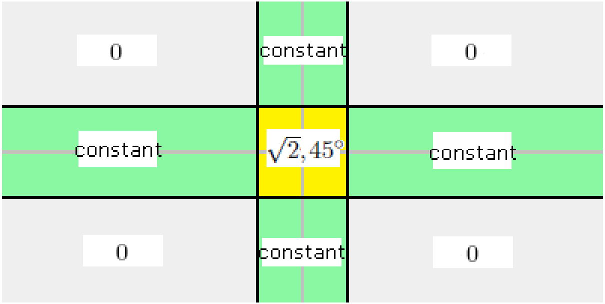
Here is a function we'll call $\gamma(z)$:



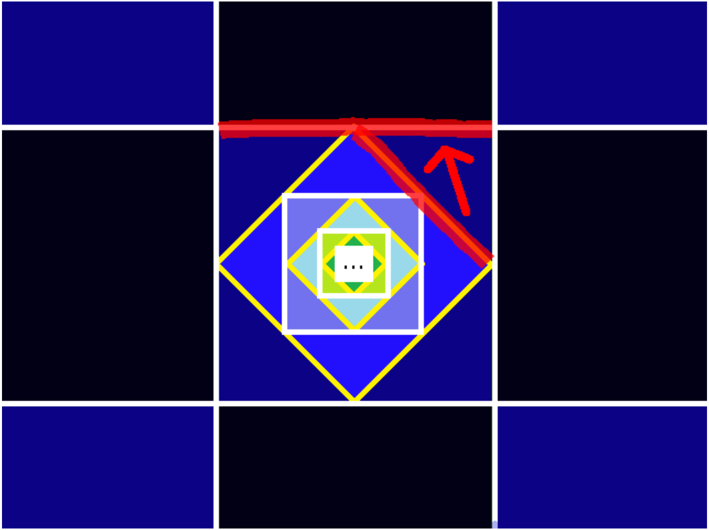
$f(z) = z\gamma(z)$

Given $z = re^{i\theta}$, $f(z)$ is described by

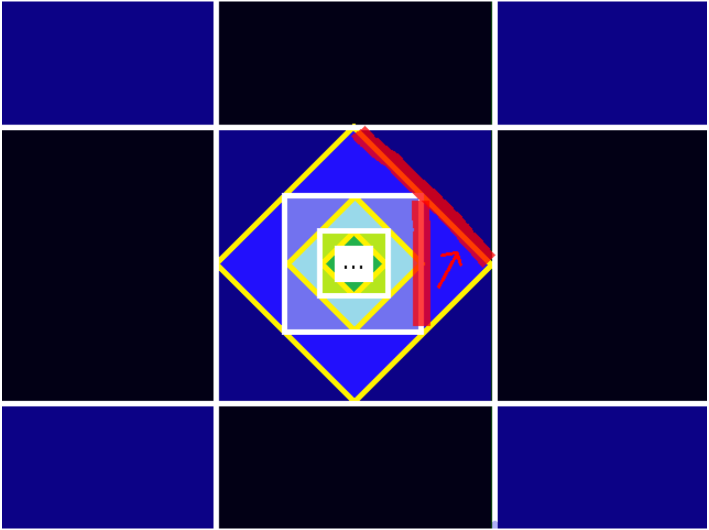
$$\begin{cases} r \mapsto \sqrt{2}r, \quad \theta \mapsto \theta + 45^\circ & \text{center box} \\ r, \theta \text{ constant} & \text{strips} \\ r, \theta \mapsto 0 & \text{elsewhere} \end{cases}$$



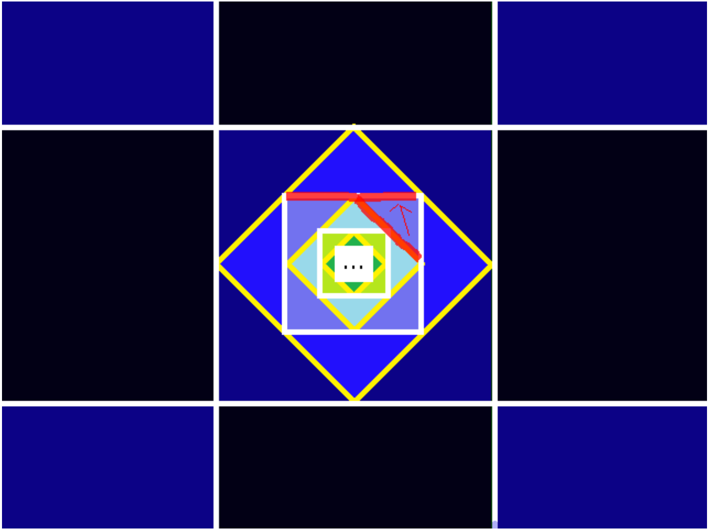
$f(z) = z\gamma(z)$



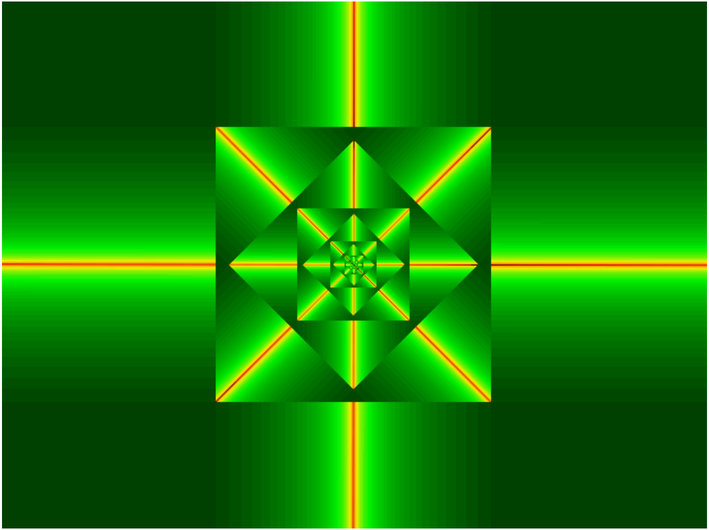
$f(z) = z\gamma(z)$



$f(z) = z\gamma(z)$



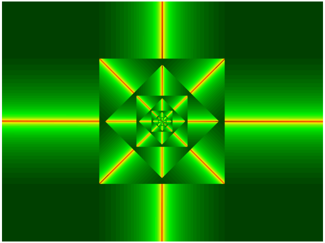
$f(z) = cz\gamma(z)$, $c = 1.1$



$c = 1.1$

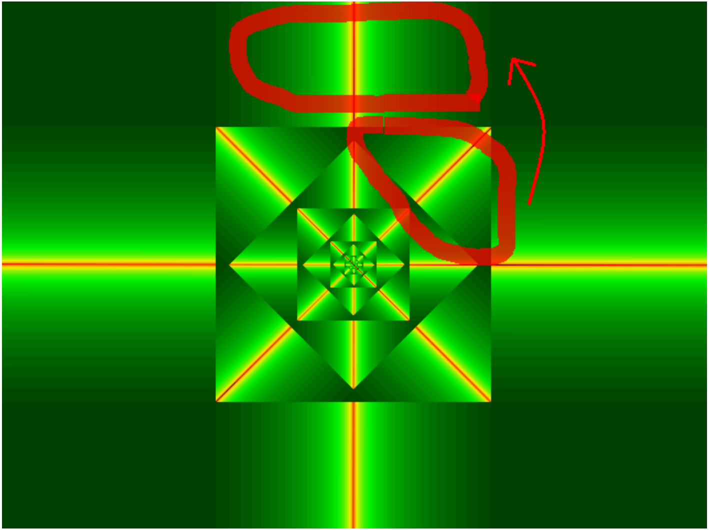
$f(z)$ is described by:

$$\begin{cases} (1.1\sqrt{2}, \quad 45^\circ) & \text{center box} \\ (1.1, \quad 0^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



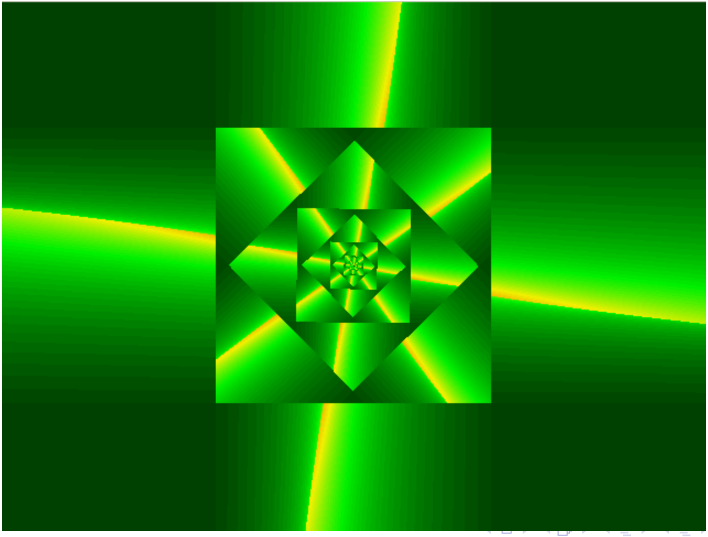
In the outside strips, the small dilation leads to slow convergence. Points within the square eventually get pushed into the outside strips.

$c = 1.1$

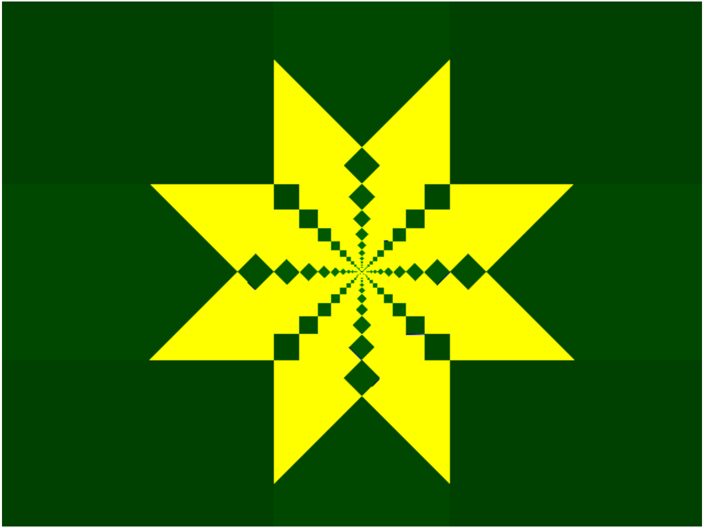


$c = 1.1 + .01i$

Adding a small imaginary term adds a bit of rotation, but no major change.



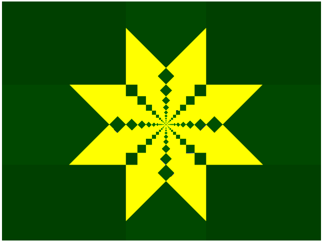
$$c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$



$c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

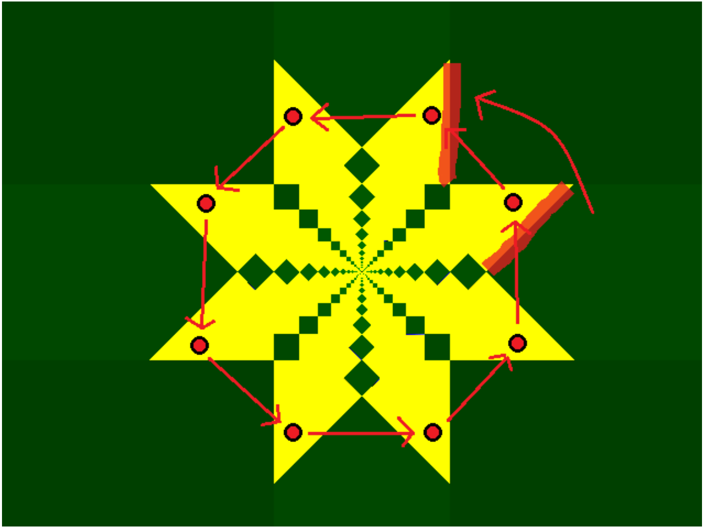
$f(z)$ is described by

$$\begin{cases} (\sqrt{2}, \quad 90^\circ) & \text{center box} \\ (1, \quad 45^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$

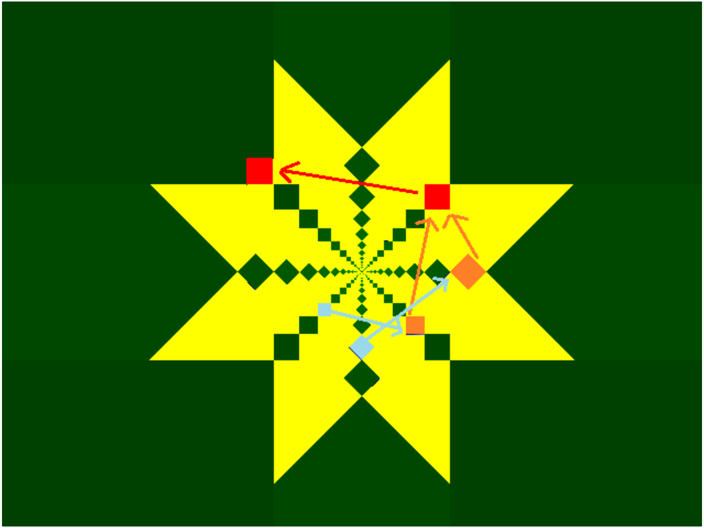


$c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Many points will cycle endlessly.

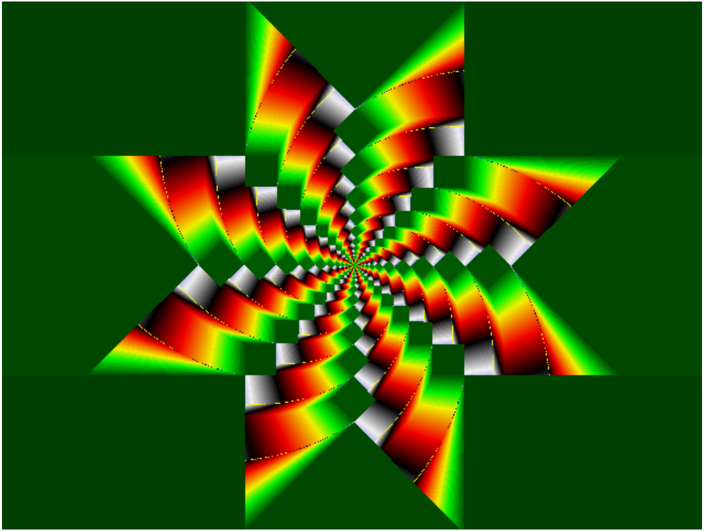


Where the green boxes and diamonds come from:



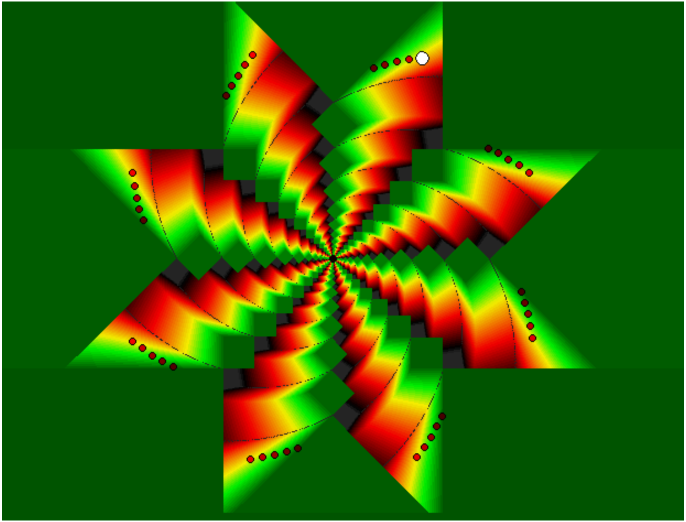
$c = .700 + .709i$

Move from $c \approx .707 + .707i$ to $.700 + .709i$.

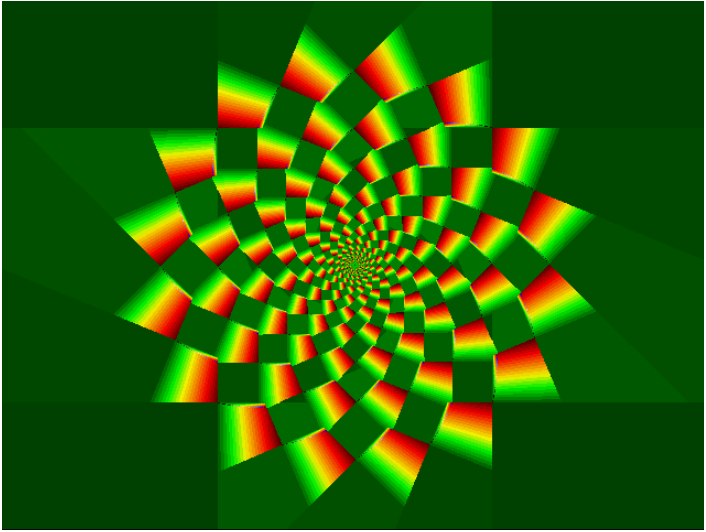


$c = .700 + .709i$

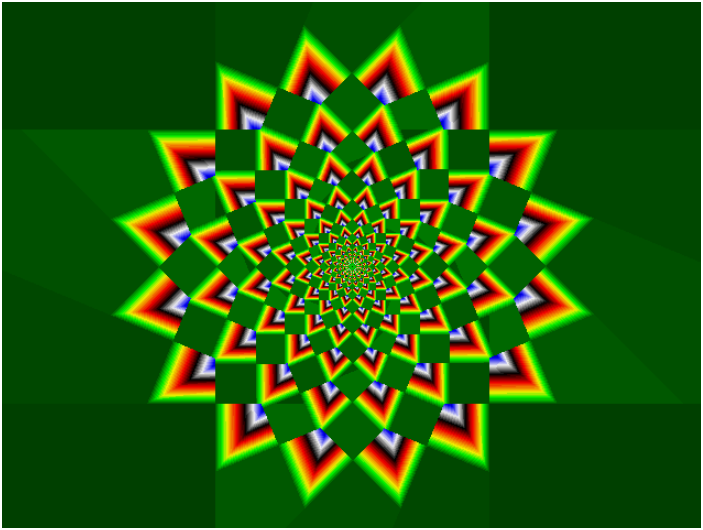
The red circles are the actual iterates. Rotation is not quite $45°$. The slight perturbation adds up.



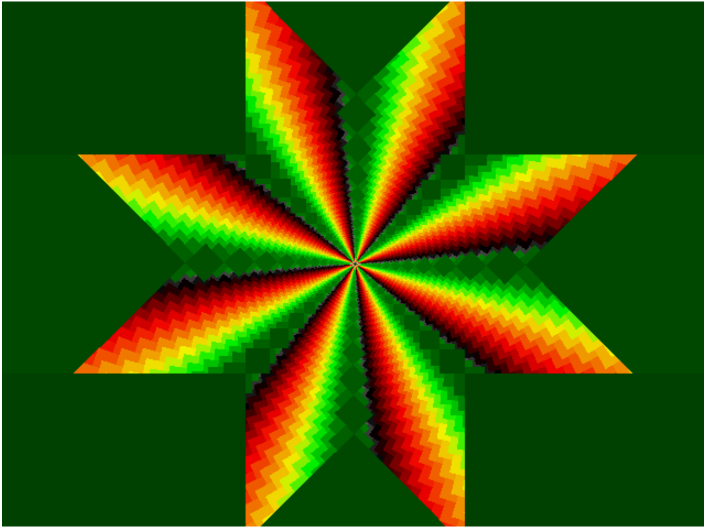
$c = .926 + .381i$



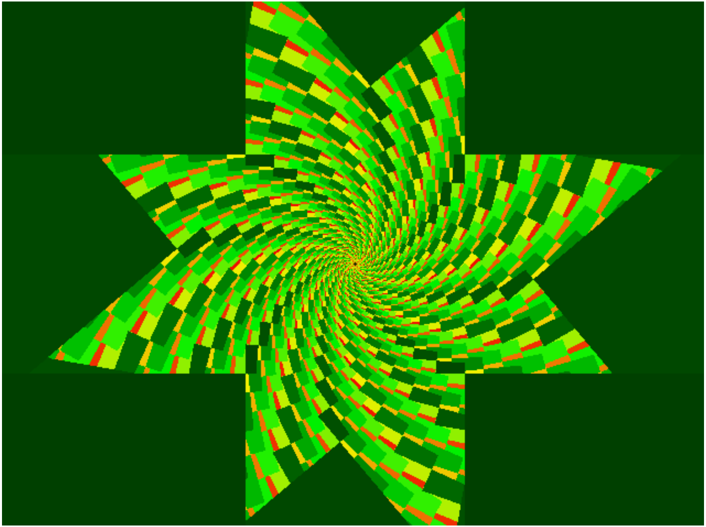
$c = .926 + .384i$



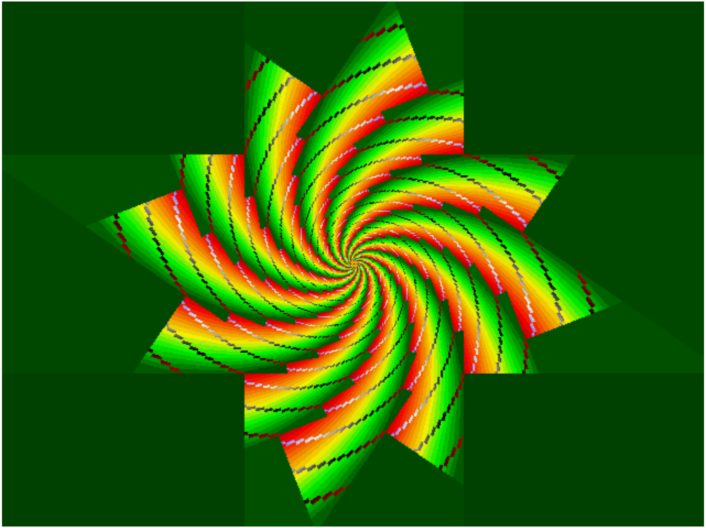
$c = .655 + .653i$



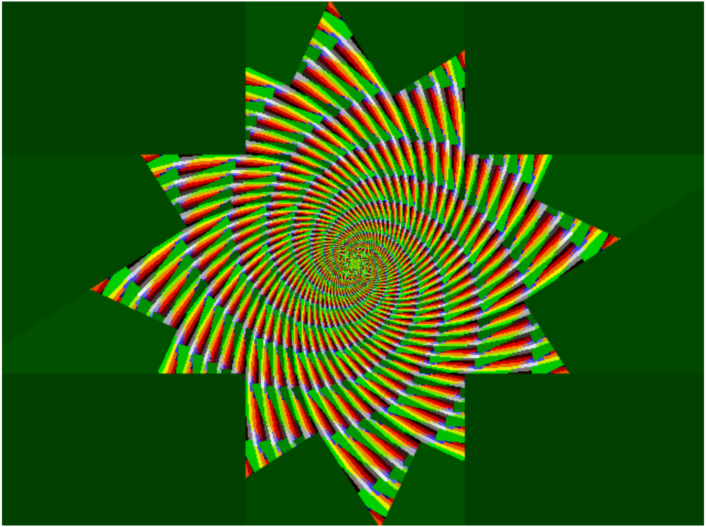
$c = .561 + .667i$



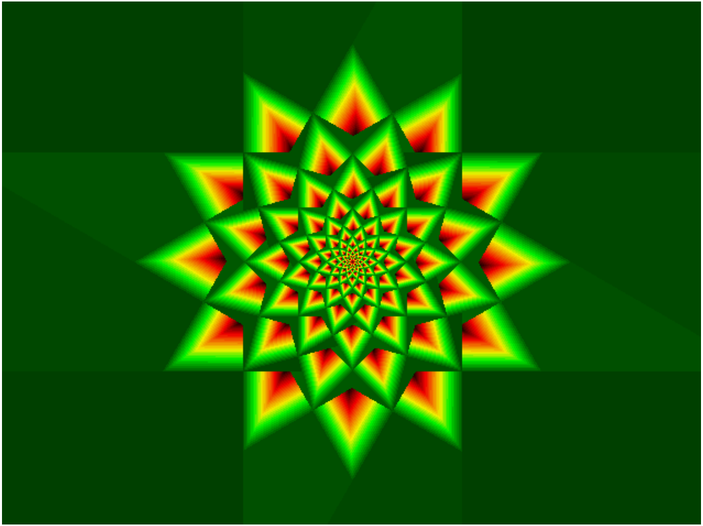
$c = .752 + .516i$



$c = .489 + .765i$

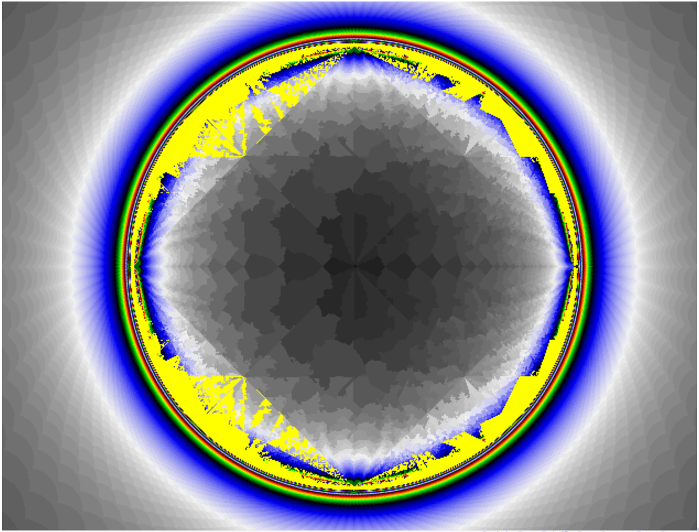


$c = .870 + .504i$



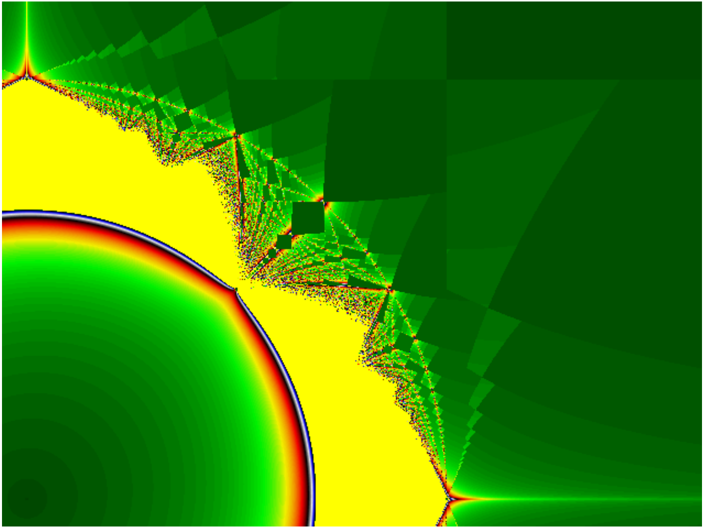
Index set

For each value of c, see what color we get when we iterate starting at $z = 1$.

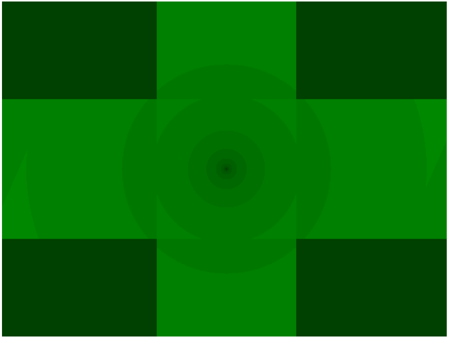
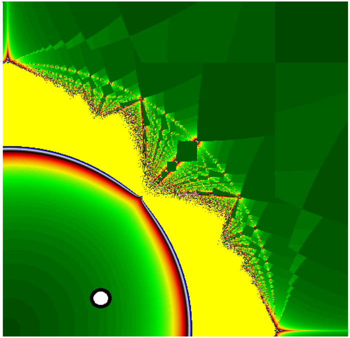


Index set close-up

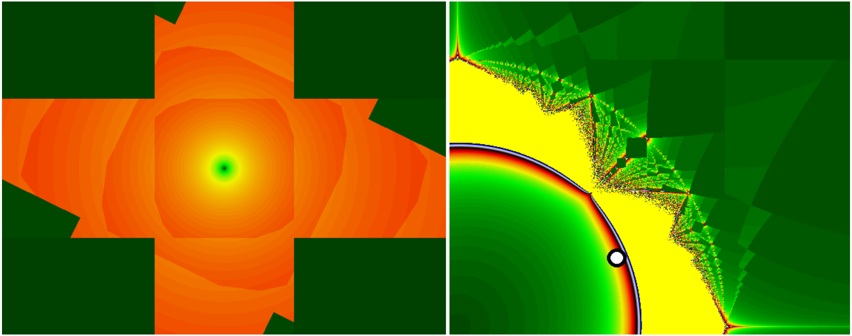
Color of $z = 1$ is somewhat representative of the entire image.



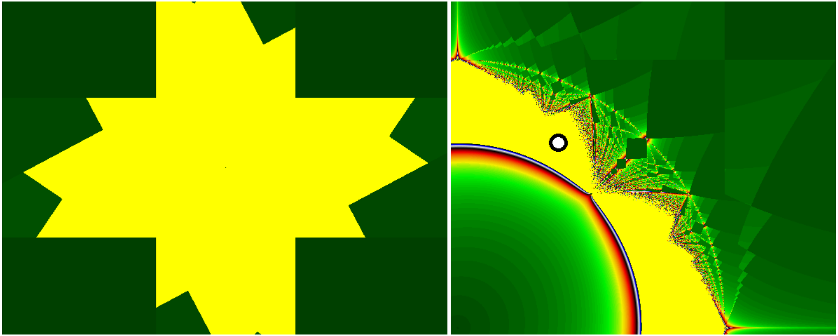
$c = .337 + .151i$

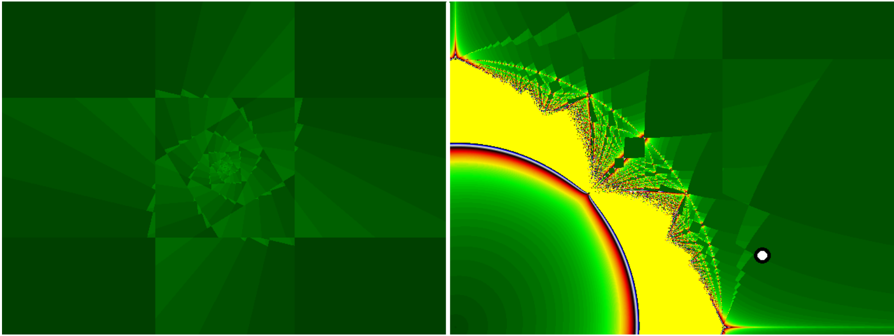
$c = .584 + .287i$



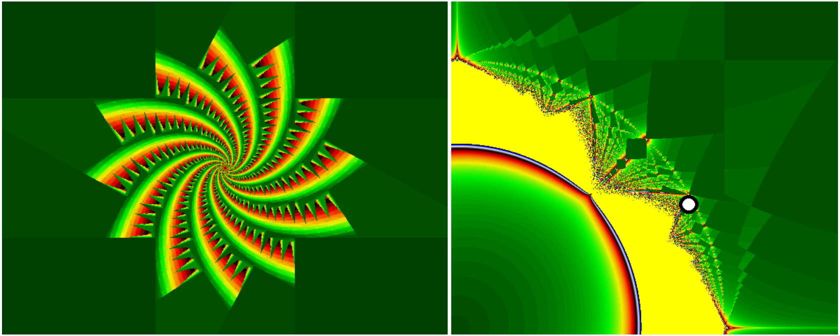
$c = .381 + .683i$



$c = .1139 + .271i$



$c = .854 + .465i$

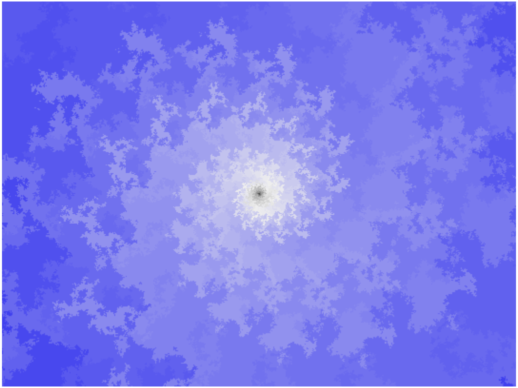


Iterating the floor function

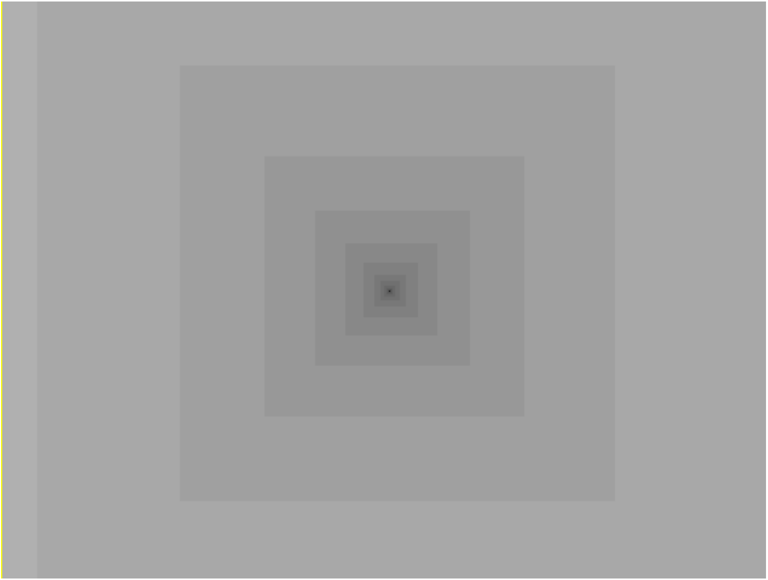
Define

$$F(z) = \lfloor x \rfloor + i \lfloor y \rfloor, \text{ where } z = x + iy.$$

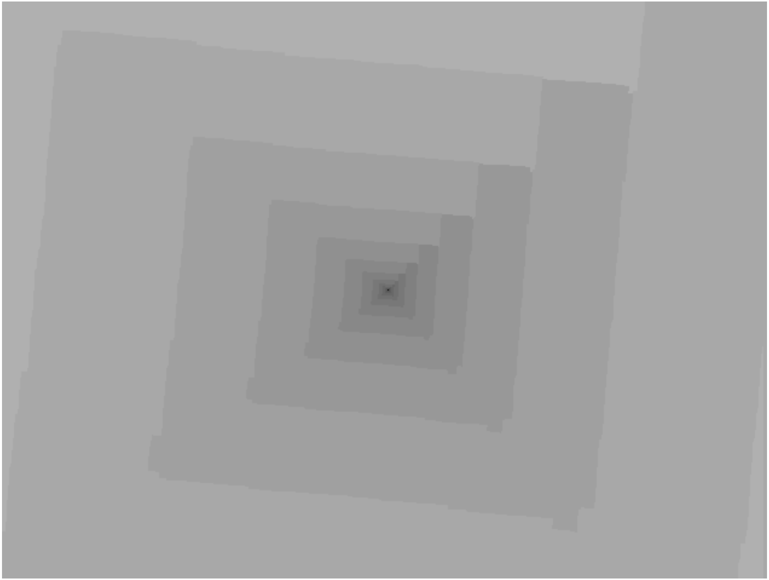
We will be iterating $cF(z)$ for various values of the constant c.



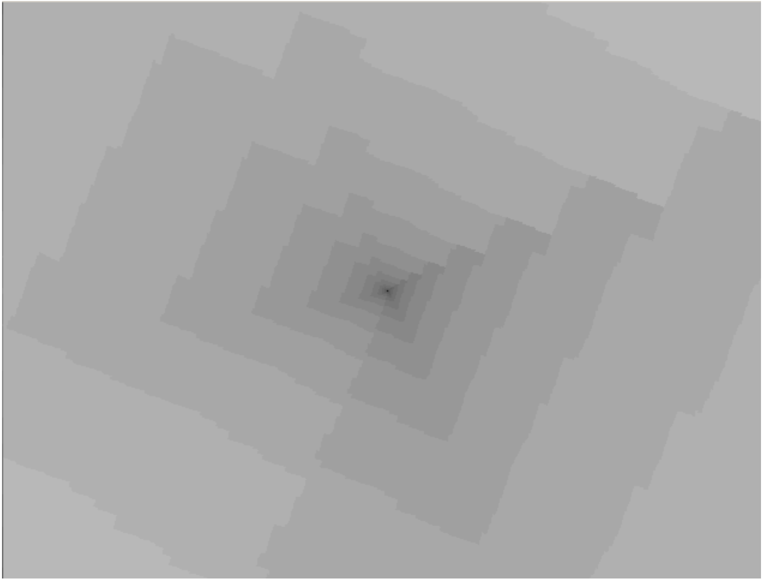
$c = .6$



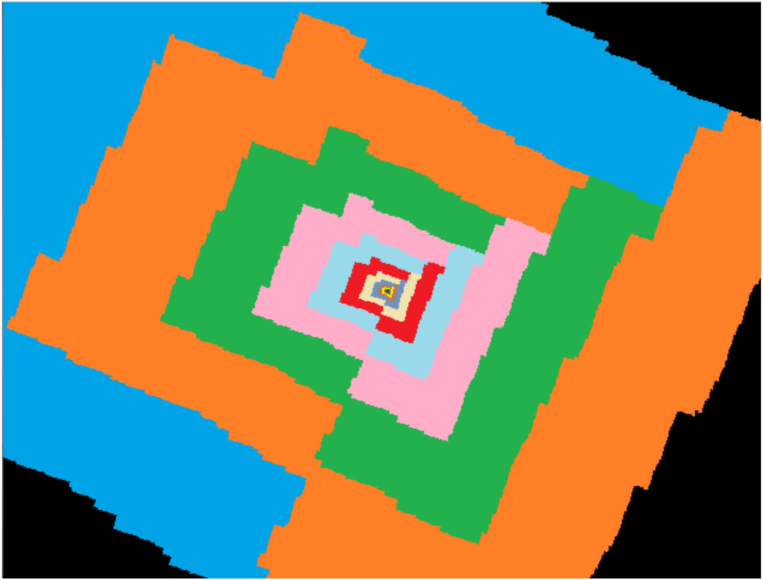
$c = .6 + .01i$



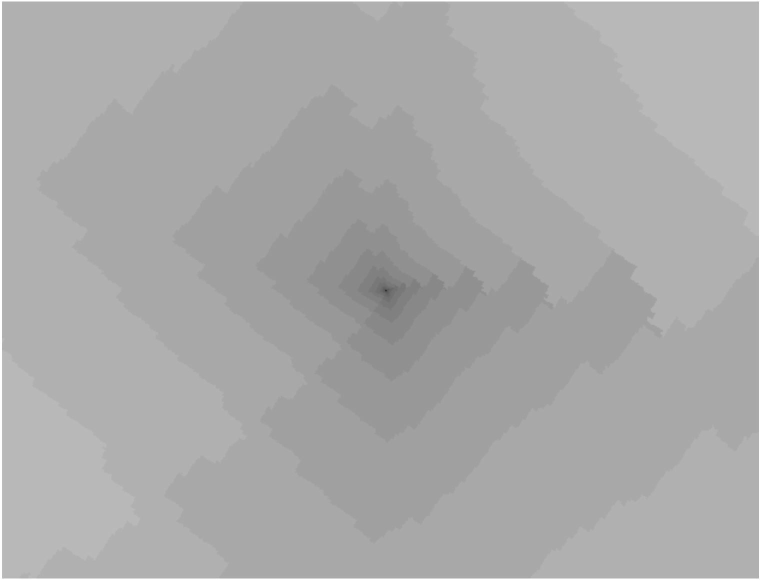
$c = .6 + .02i$



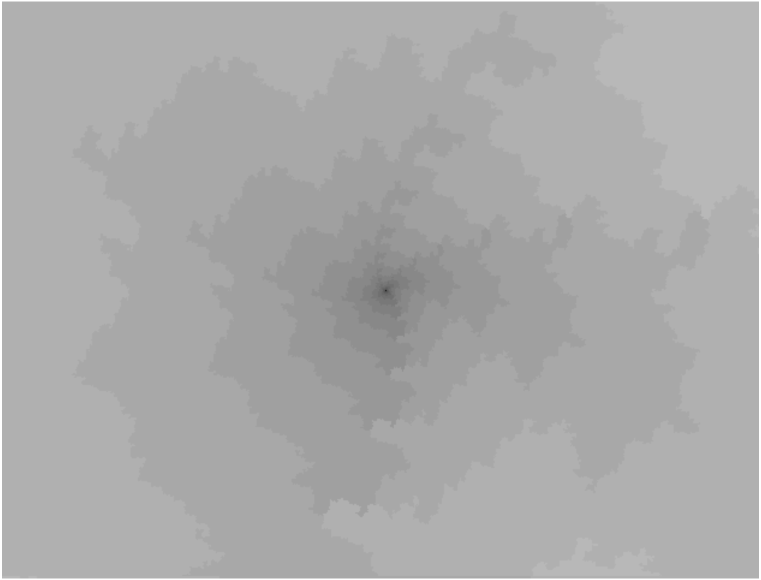
$c = .6 + .02i$ false color



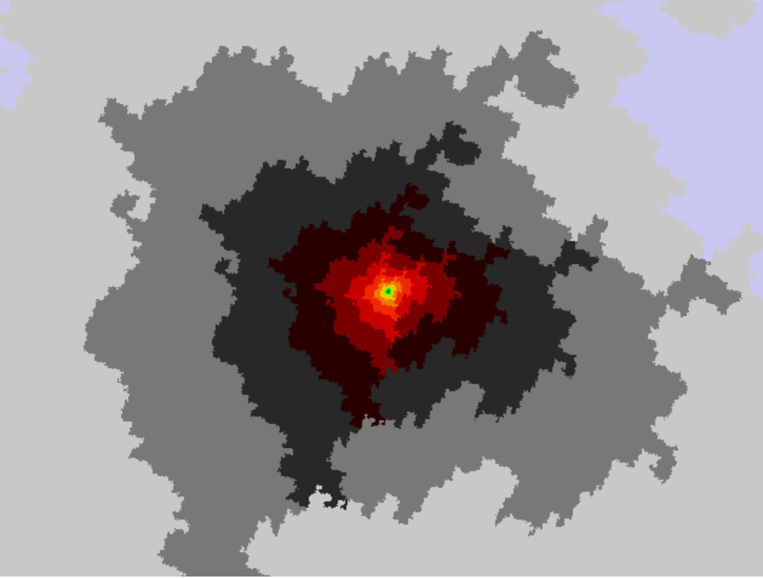
$c = .6 + .03i$



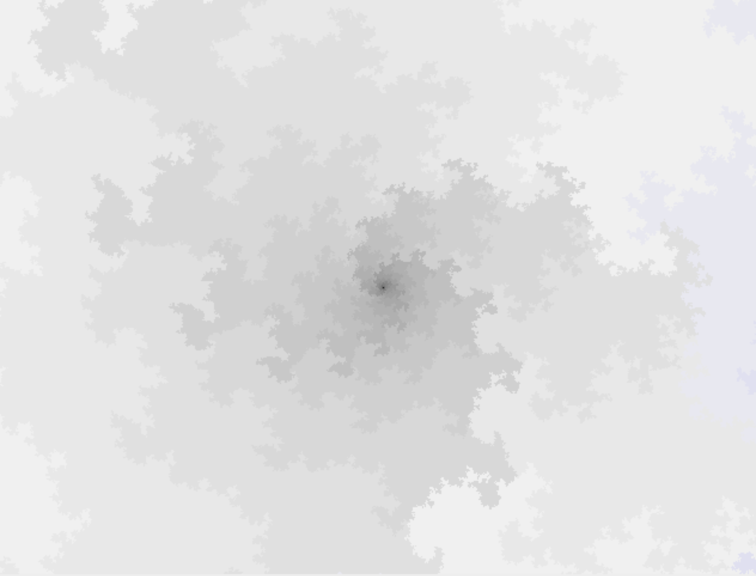
$c = .6 + .1i$



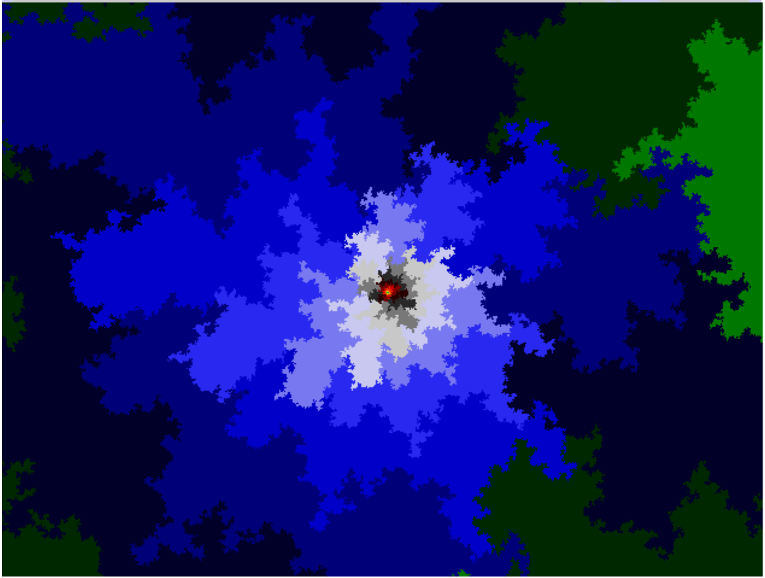
$c = .6 + .1i$ sharper gradient



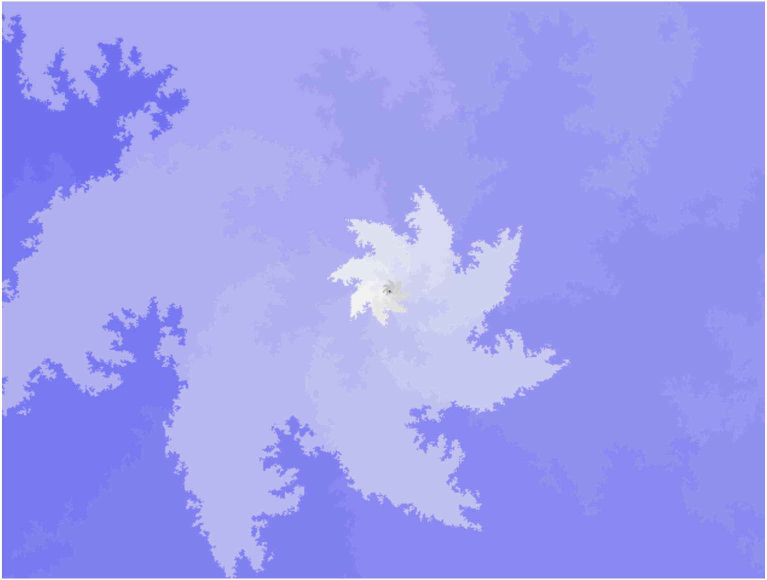
$c = .6 + .3i$



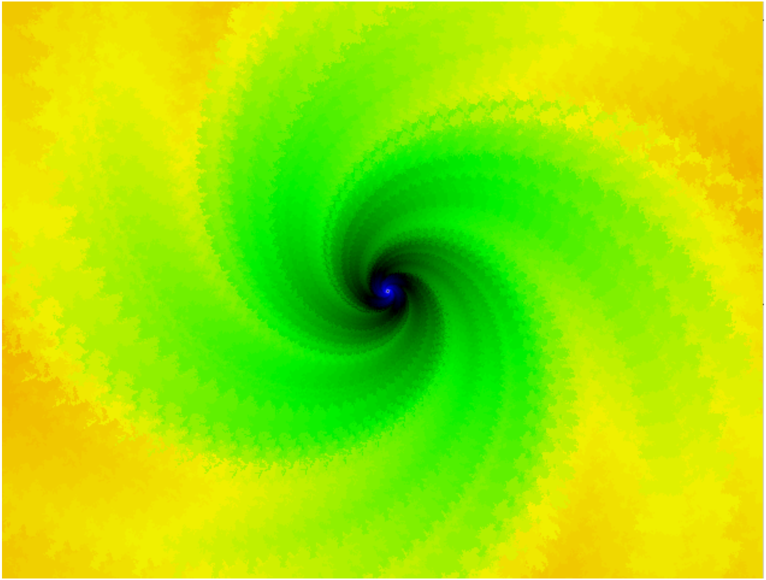
$c = .6 + .3i$ sharper gradient



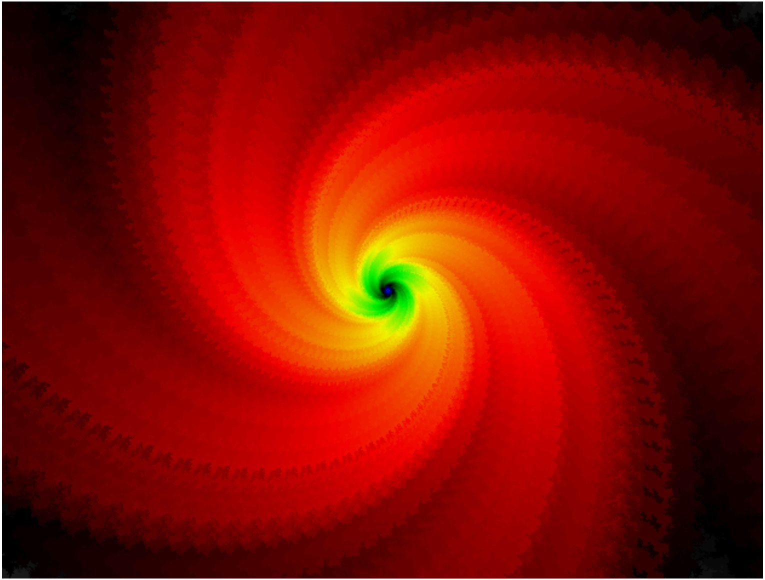
$c = .51 + .56i$



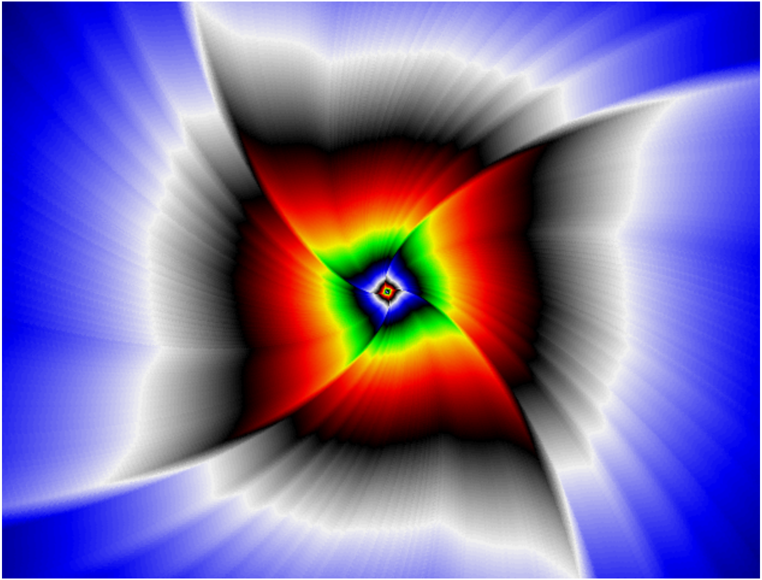
$c = .94 + .09i$



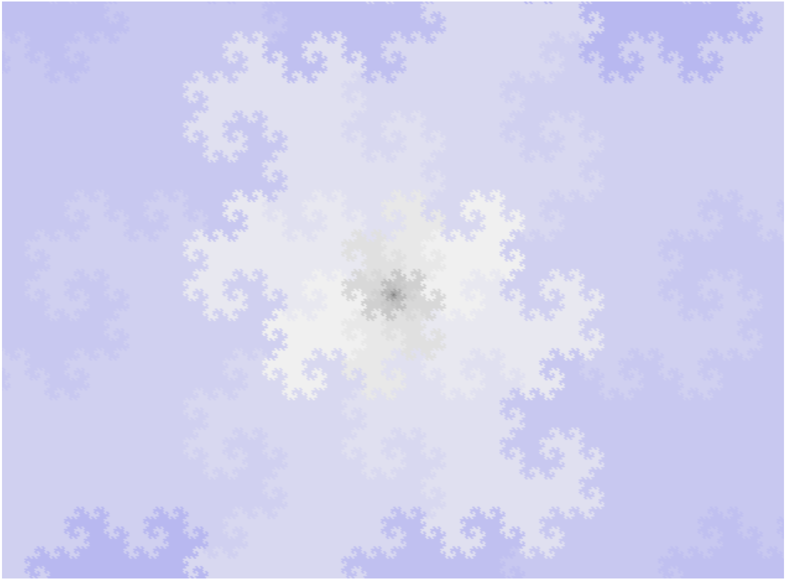
$c = .96 + .06i$



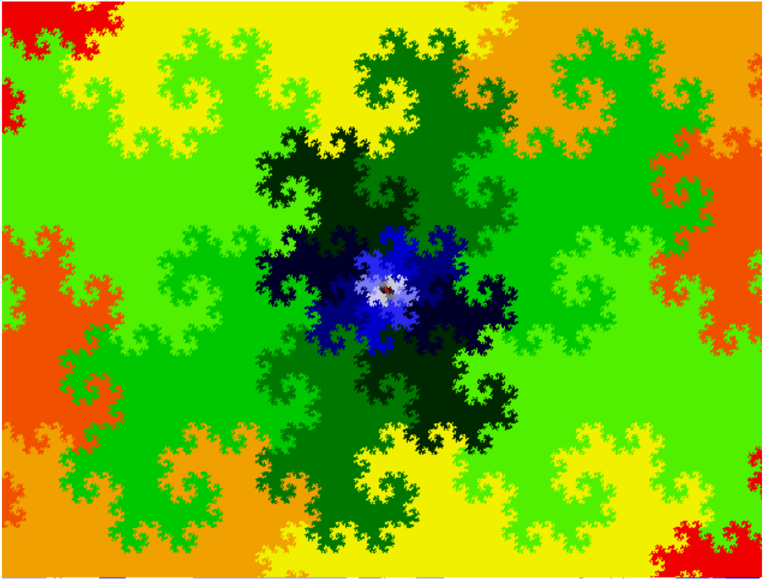
$c = .99 + .01i$



$c = .5 + .5i$



$c = .5 + .5i$



$c \approx .5 + .5i$

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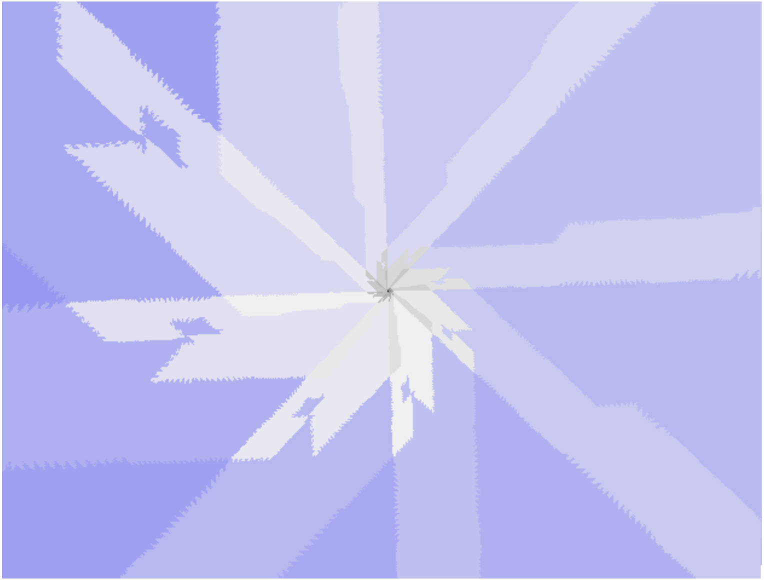
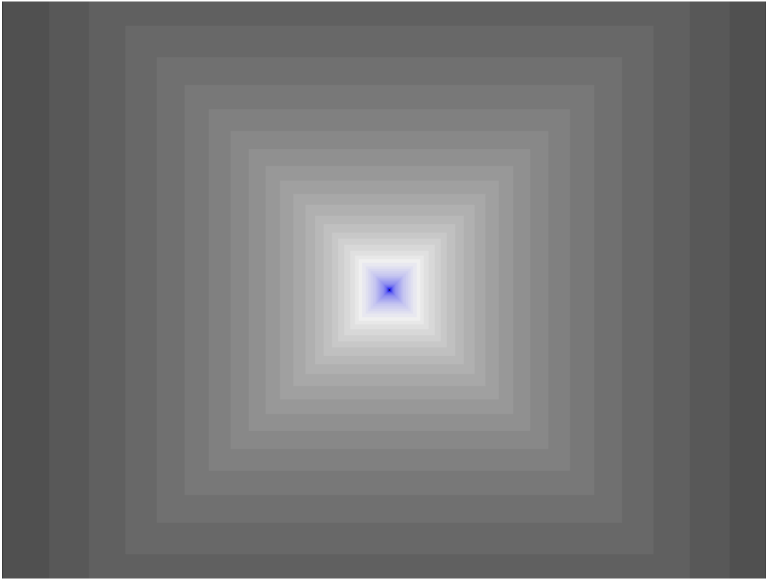
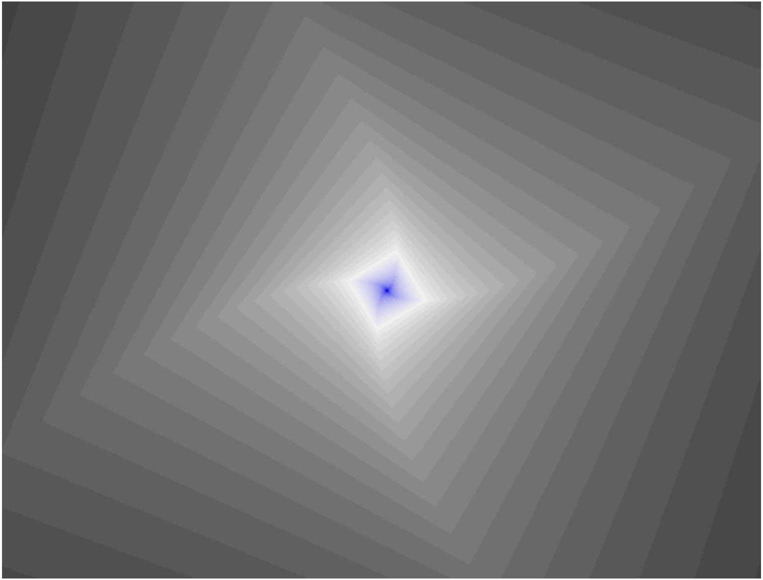


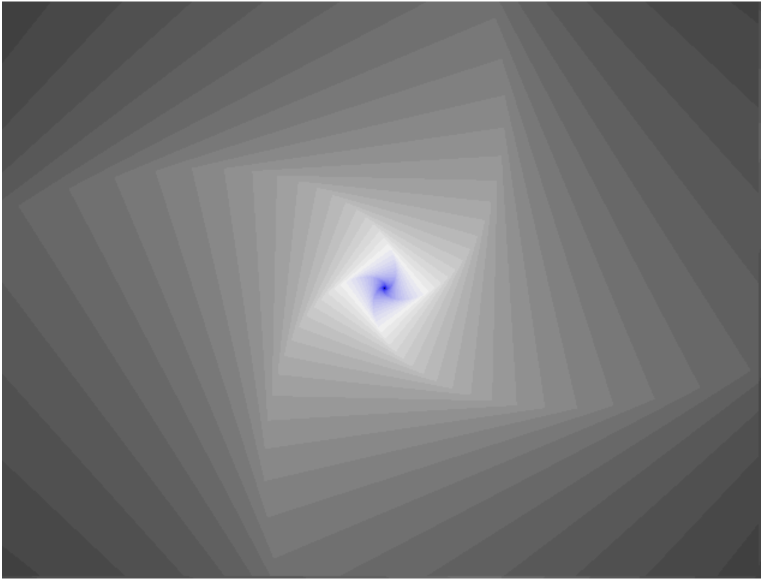
$c = 1.14$



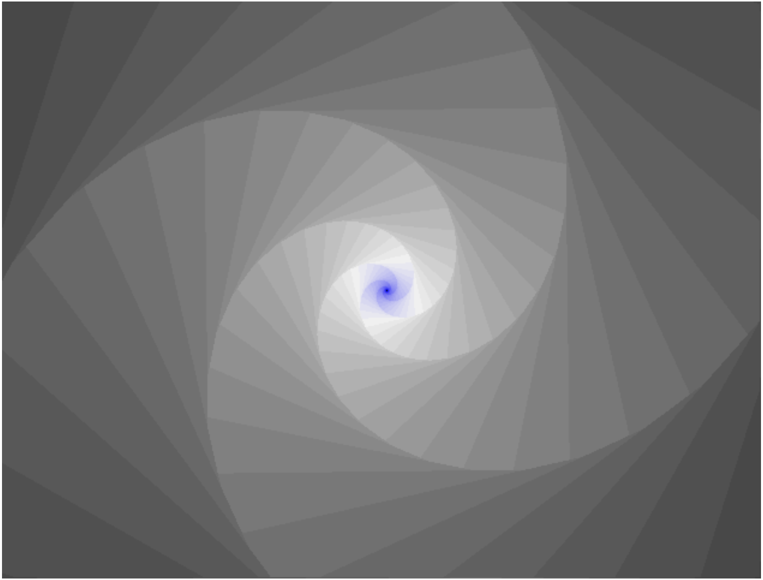
$c = 1.14 + .04i$



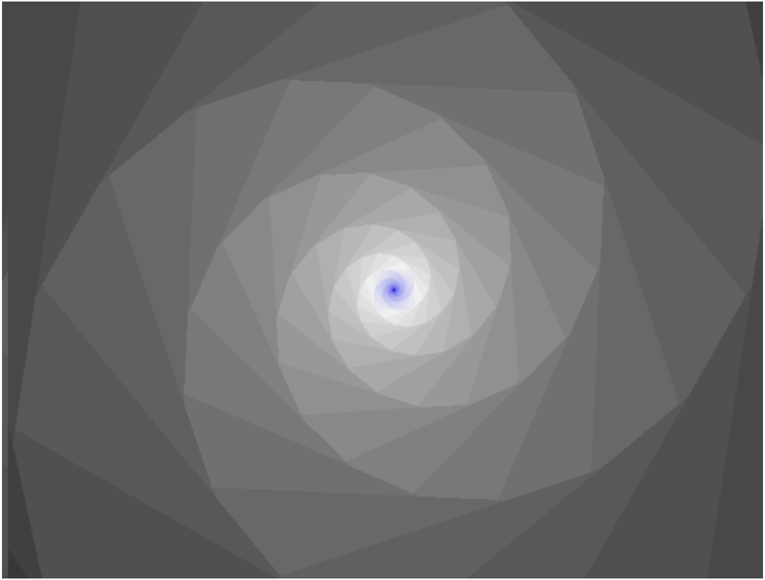
$c = 1.13 + .1i$



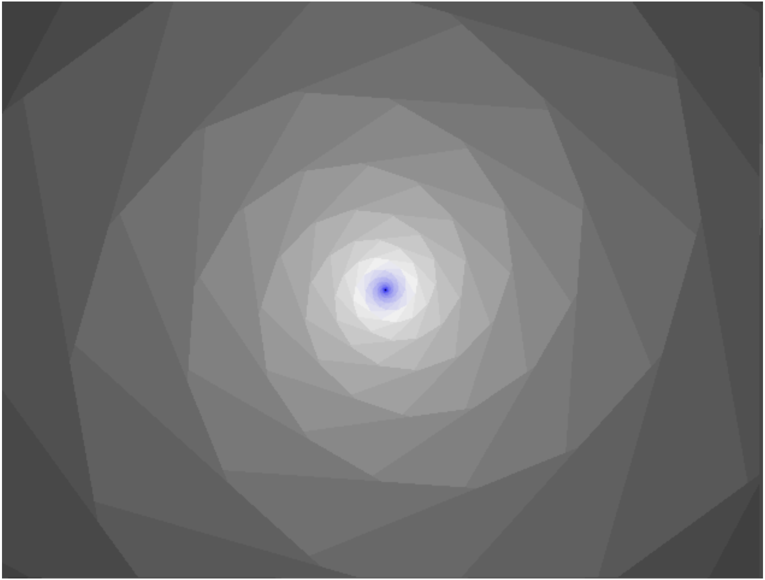
$c = 1.12 + .24i$



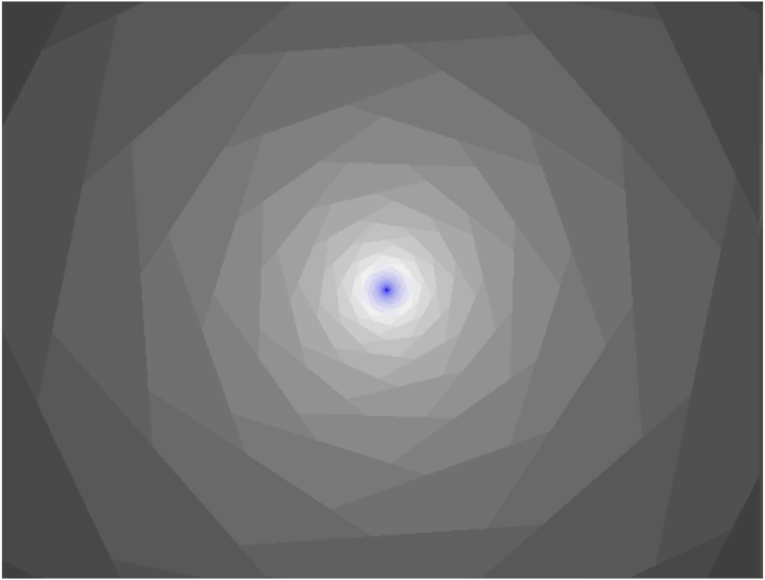
$c = 1.07 + .41i$



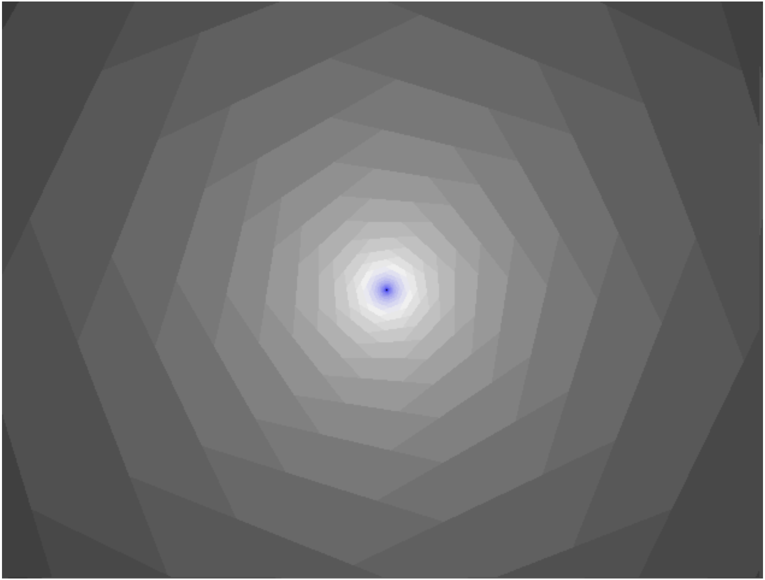
$c = 1.02 + .5i$



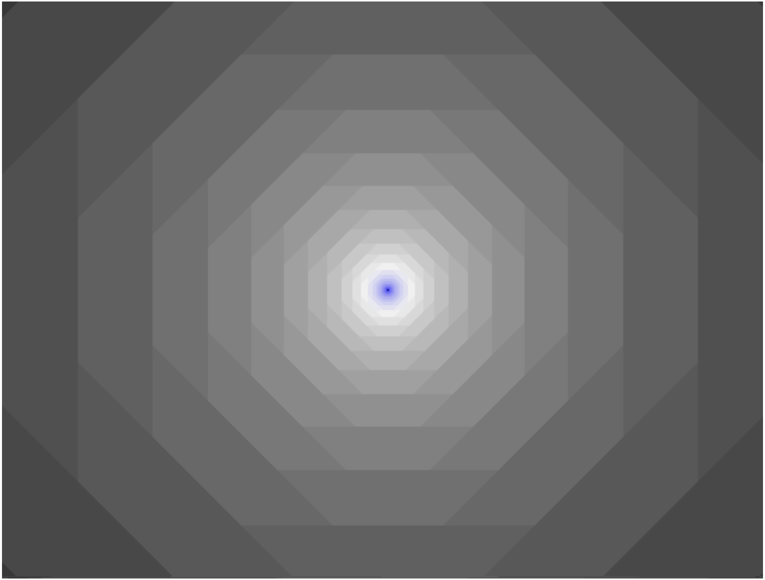
$c = .91 + .69i$



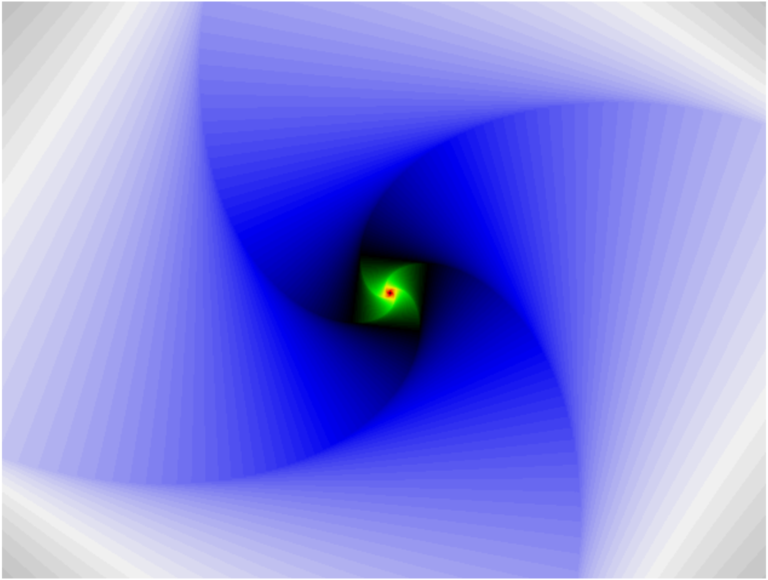
$c = .84 + .78i$



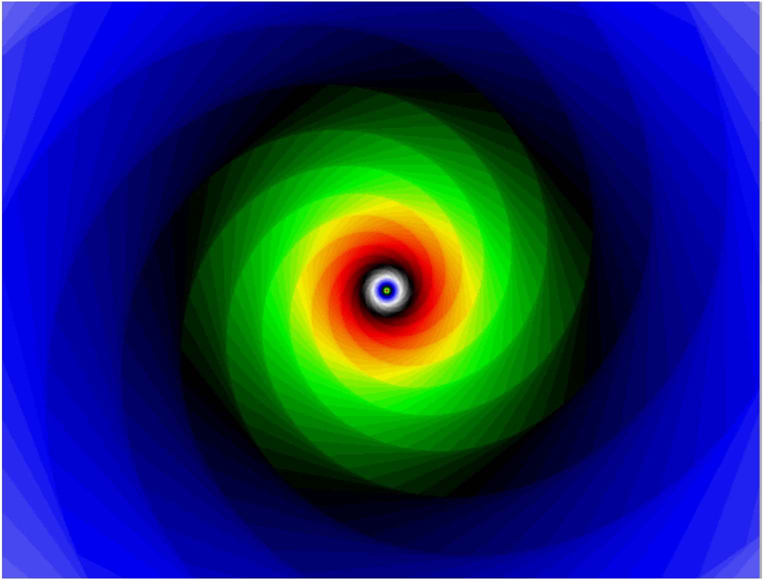
$c = .81 + .81i$



$c = .04 + 1.04i$

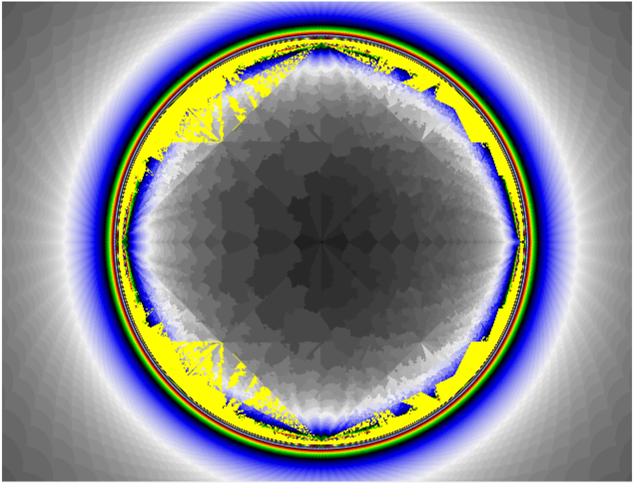


$c = .68 + .77i$

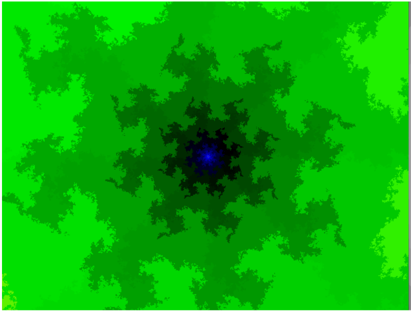


Index set

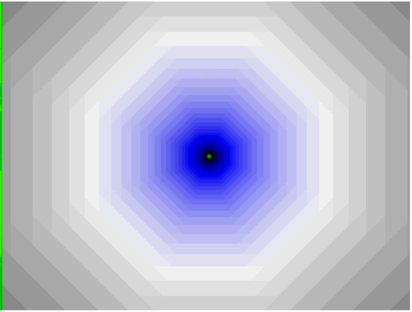
Look at what happens to the point $(50, 50)$ under iteration for various values of c.



Inside unit circle vs outside

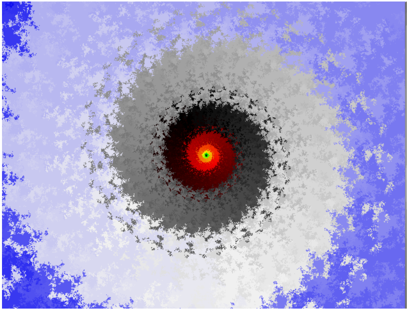


.65+.65i (inside)

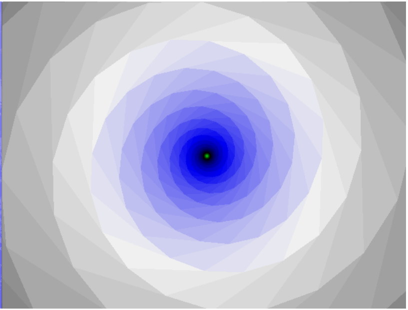


.75+.75i (outside)

Inside unit circle vs outside



.91+.31i (inside)



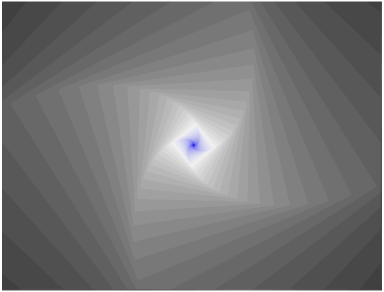
1+.34i (outside)

Outside the unit circle

Outside: Iterates attracted to ∞.

Iteration determined by relatively simple interaction between:

- Rotation from multiplying by complex values of c
- Floor function
- The norm used. Iterates "converge" to ∞ when $|x| > 10^6$ or $|y| > 10^6$. Using the Euclidean norm removes all interesting behavior.

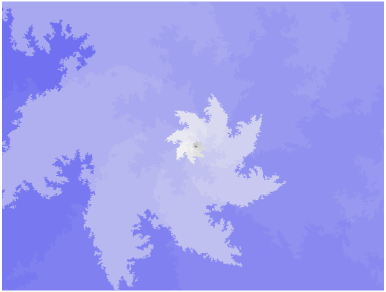


Inside the unit circle

Inside: Iterates attracted to various fixed points.

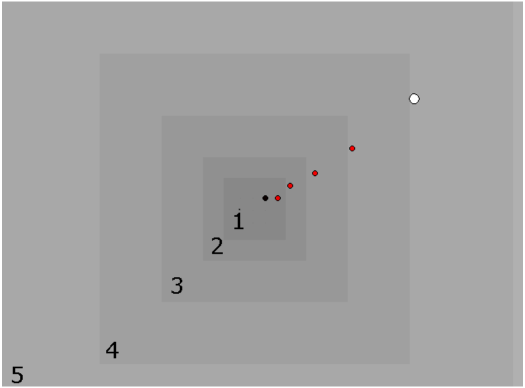
Iteration determined by

- Rotation from multiplying by complex values of c
- Floor function



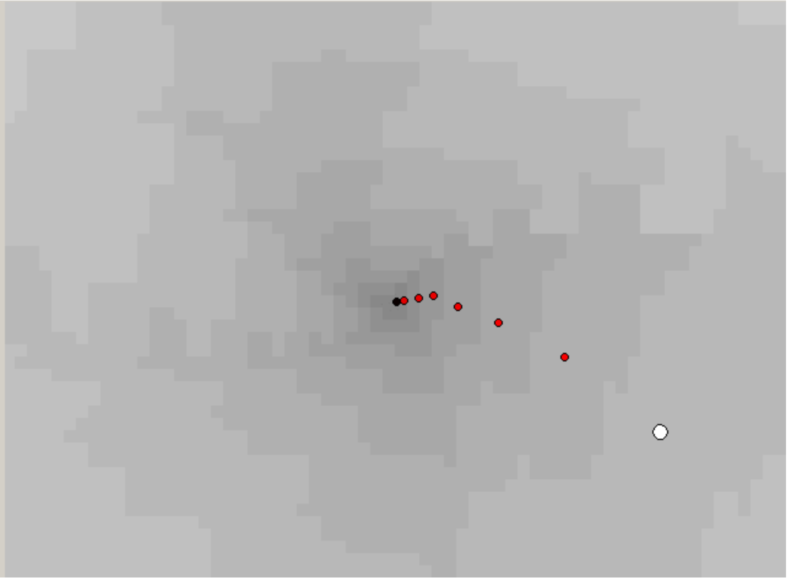
Closer look at $c = .6$

Nine fixed points: all the points of
$\{-1.2, -.6, 0\} \times \{-1.2, -.6, 0\}$

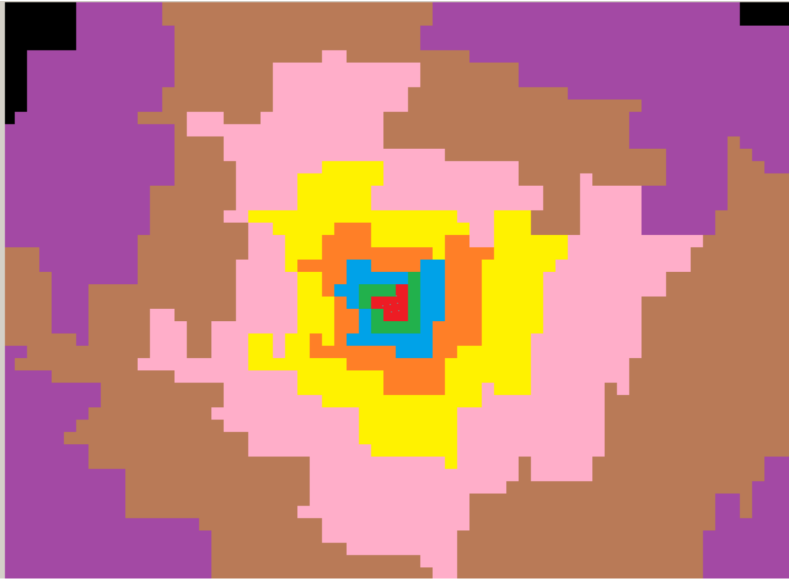


Box n = {points mapping to fixed point in n iterations}

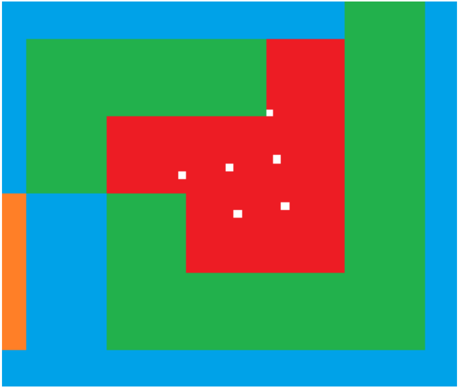
Closer look at $c = .6 + .1i$



Closer look at $c = .6 + .1i$ in false color

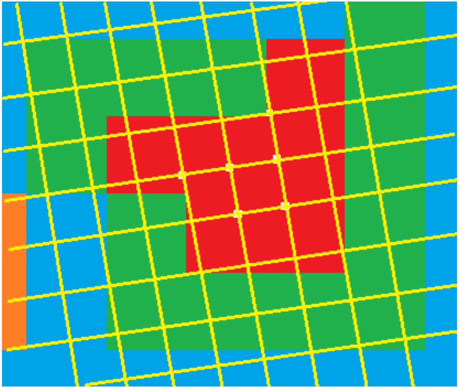


Fixed points of $c = .6 + .1i$



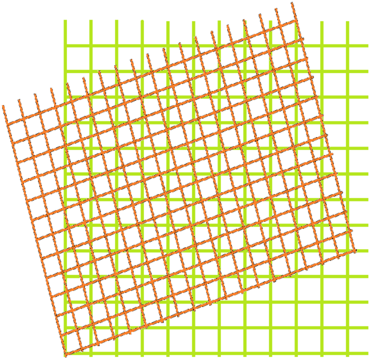
Fixed points:
$(.1, -.6), (-.5, -.7), (.2, -1.2), (0, 0), (-1.1, -.8), (-.4, -1.3)$

Slanted grid for $c = .6 + .1i$



All iterates constrained to move along slanted grid (slopes 1/6 and -6).

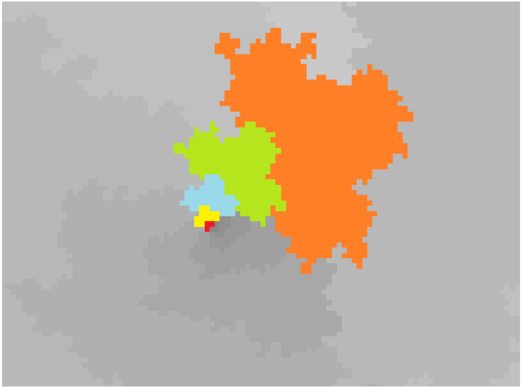
Slanted grid for $c = .6 + .1i$



Interaction between rectangular grid induced by floor and slanted grid induced by complex multiplication

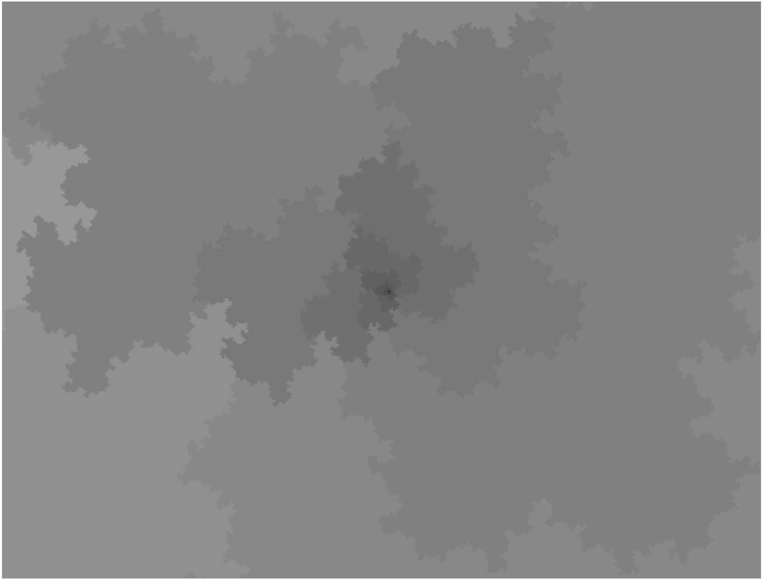
Can describe this iteration purely in terms of rotations, dilations, and "snapping to the grid."

Closer look at $c = .43 + .23i$

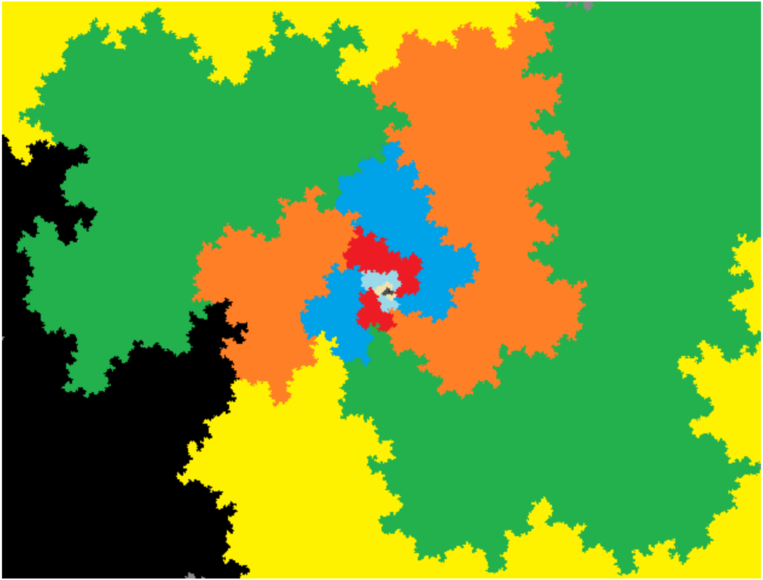


Each colored segment is a “copy” of one before it, becoming more complex in a fractal-like way.

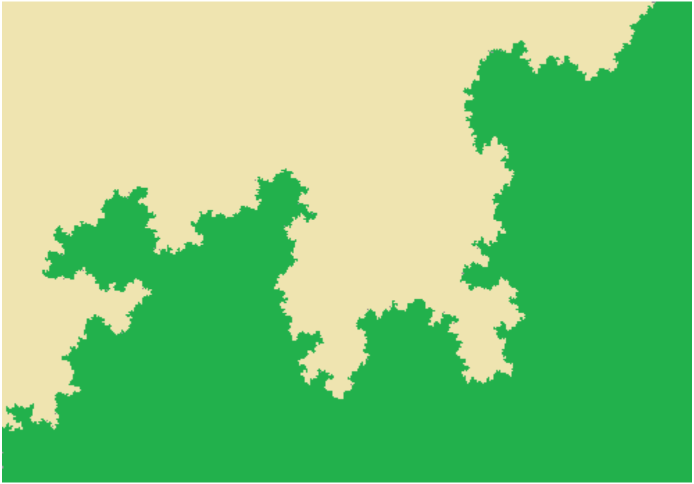
$c = .43 + .23i$



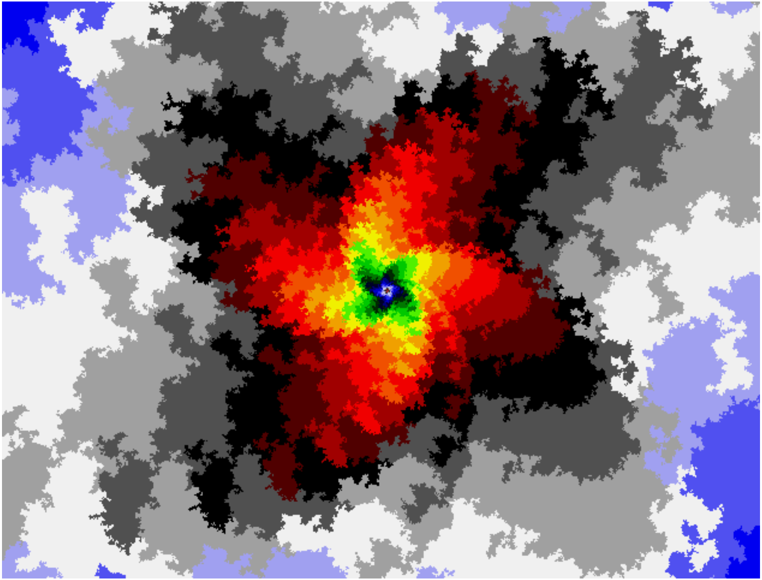
$c = .43 + .23i$ false color



Far zoom out of a section from $c = .43 + .23i$



$c = .78 + .14i$ sharper gradient



$c = .64 + .34i$ sharper gradient