

Probability Questions from the Game Pickomino

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July 28, 2017



a.k.a. *Heckmeck am Bratwurmeck*

Created by Reiner Knizia

Published by *Zoch zum Spielen* and Rio Grande Games

2005

How to Play

- 8 dice, each with numbers 1-5 and a worm (worth 5)



- Tiles with costs 21-36, worth 1, 2, 3, or 4 points



How to Play, Continued

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- Reroll the dice not taken and repeat until (a) you have enough to take a tile or (b) you are unable to take any dice (because all the dice in a reroll match ones already taken)
- Key rule: You need at least one worm in order to take a tile.

Example Turn, First Roll

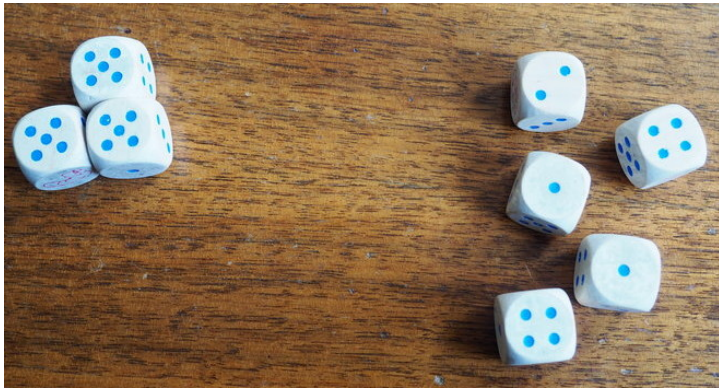


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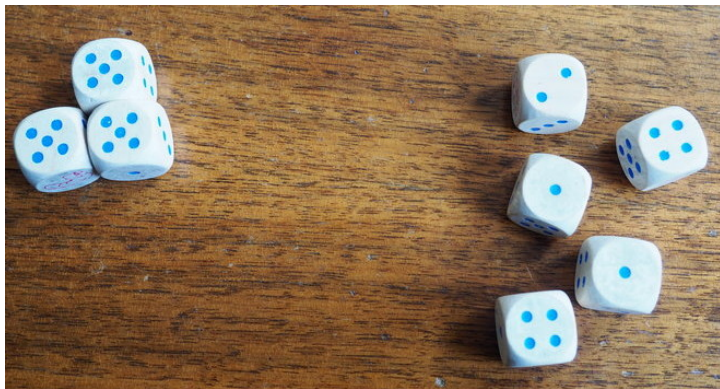


Take the three 5s.

Example Turn, Second Roll

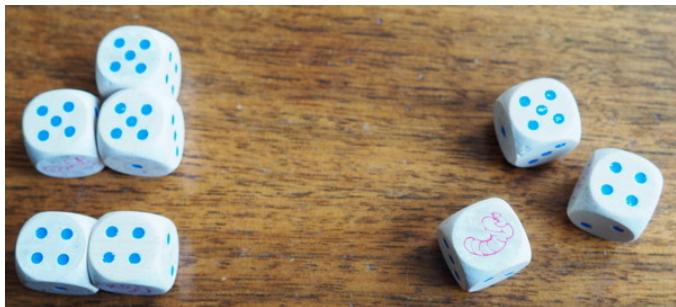


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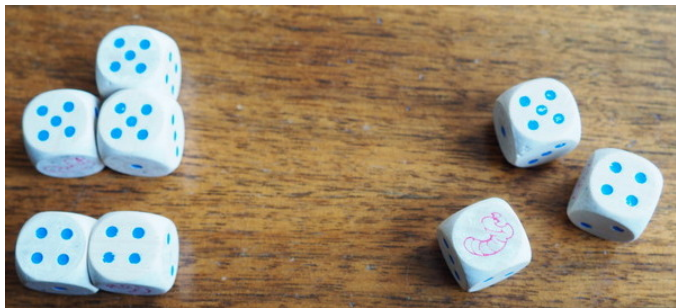


Take the 4s. We're up to 23 points, but no worms, so we can't take anything.

Example Turn, Third Roll



Example Turn, Third Roll



Take the worm. We're up to 28 points. Let's grab the 28 tile. It's worth 2 points.



How to Play, Continued

- If you fail, you lose the tile you most recently acquired. It goes back to the center and the highest numbered tile is turned over (unless the tile returned was the highest).
- Fun note: if you get exactly the number of points as an opponent's most recently taken tile, you can steal it.
- Game ends when all tiles are taken. Winner is one with most points (total worms on tiles).



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- This is a nice example for any class covering basic probability.

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- So if we are holding no tiles or a 1-tile, it might make sense to go for more. Losing that 1-tile might expose a high tile...

A Simple Question



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- The probability is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$.

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- We had a 5 and two worms and our next roll was all 5s and worms!
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- $\left(\frac{2}{6}\right)^5 = \frac{1}{243}$
- Seems like it happens to me more often than that!

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- These are nice examples of binomial probabilities.

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- So, in total, a little under a 40% chance.

Simulation (because I don't trust my calculations)

```
from random import randint
count = 0
n = 100000
for i in range(n):
    d1 = randint(1,6)
    d2 = randint(1,6)

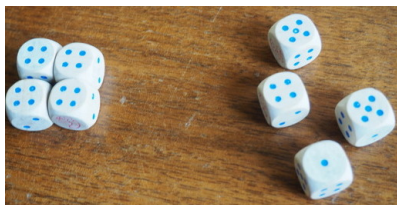
    if d1 in [4,5] and d2 in [4,5] or (d1==d2 and d1!=6):
        continue

    if d1 == 6 or d2 == 6:
        count += 1
        continue

    if randint(1,6) == 6:
        count += 1

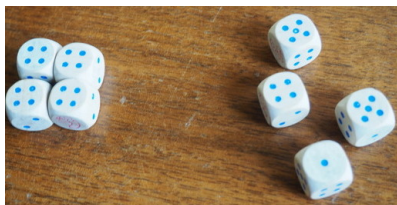
print (count/n * 100)
```

When Does It Make Sense to Take Something Small?



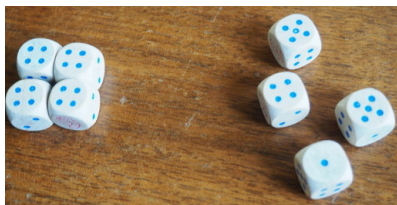
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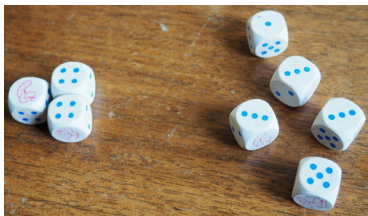
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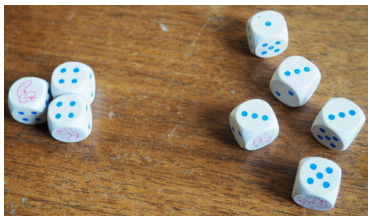
- In the roll above it might make sense to take the 1 instead of the two 5s.
- Why? It gives you a better chance of getting a worm.
- Answer, though, would depend on what tiles are available.

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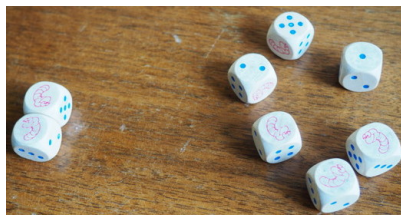
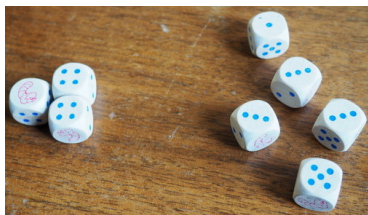
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- An interesting project would be to identify when it would make sense to take a single 1 or 2 instead of bigger things.
- Above on left, if you need a lot of points, the 1 might be the best option.
- On the right, taking a 5 would seriously limit your future points.

Questions for the First Roll



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- In general, n worms versus $n + k$ 5s?
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- Three 4s versus 1 worm?
- This would make another nice project.

More Questions for the First Roll



- Is it worth taking all those 5s? It seriously limits your chances of getting a worm.

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- What about four 4s to start?

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- I once played a two-player game, where between the two of us, we took all but one tile (and only lost that on the second to last turn). We only failed to take a tile once in all the turns.

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- How unusual was that?
- Tricky to answer because of the possibility of stealing tiles.

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- What is the probability that someone can take the 36 tile?
- What is the probability that you at least take the 21 tile? Or at least x ?

Things People Have Done

- Cardon, Chetcuti-Sperandio, Delorme, and Lagrue have a few articles on modeling Pickomino as a Markov Chain.
- See, e.g., *A Markovian Process Modeling for Pickomino*, Proceedings of the 7th International Conference on Computers and Games, in Lecture Notes in Computer Science, pp. 199-210, Springer-Verlag, 2010.

We provide in this paper original solutions to both problems: we provide (1) a compact representation of states and (2) a constructive method to compute the probability distributions, based on the partitioning of the space of roll results depending on a set of marked values. Finally, we show the efficiency of the proposed method through numerous experimental results: it turns out to be impressive compared to previous programs we developed.

Another article by the same authors: *Determination and evaluation of efficient strategies for a stop or roll dice game: Heckmeck am Bratwurmeck (Pickomino)*, Computational Intelligence and Games, 2008. CIG '08. IEEE Symposium On Computational Intelligence and Games.

We propose then an algorithm using a Monte-Carlo method to evaluate probabilities of dice rolls and the accessibility of resources. By using this tactical computing in different ways the programs can play according to the stage of the game (beginning or end). Finally, we present experimental results comparing all the proposed algorithms. Over 7,500,000 matches opposed the different AIs and the winner of this contest turns out to be a strong opponent for human players.

Thomas ten Cate at the blog Frozen Fractal

(<http://frozenfractal.com/blog/2015/5/3/how-to-win-at-pickomino/>)
created a computer player, using a dynamic programming approach to traverse the game tree. It seems to be reasonably effective. His code is on GitHub.

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- There are various questions that would make good projects.
- It would be interesting to program a computer to play the game, and compare various strategies (e.g. Markov vs. searching probability tree).

Thanks!

