Smalltalk 3/7/13

e

Brian Heinold

What is this sequence approaching?

1^1
1.1^{10}
1.01^{100}
1.001^{1000}
1.0001^{10000}
$\ldots$

It is approaching $e \approx 2.718$

$$
\begin{aligned}
&1^1 = 1 \\
&1.1^{10} = 2.593\ldots \\
&1.01^{100} = 2.704\ldots \\
&1.001^{1000} = 2.716\ldots \\
&1.0001^{10000} = 2.718\ldots \\
&\ldots
\end{aligned}
$$

A more general formula

The previous is a special case of

$$e = \lim \left(1 + \frac{1}{n}\right)^n$$

A more general formula

The previous is a special case of

$$e = \lim \left(1 + \frac{1}{n}\right)^n$$

$\left(1 + \frac{1}{2}\right)^2$
$\left(1 + \frac{1}{3}\right)^3$
$\left(1 + \frac{1}{4}\right)^4$
$\left(1 + \frac{1}{5}\right)^5$

A more general formula

The previous is a special case of

$$e = \lim \left(1 + \frac{1}{n}\right)^n$$

$$\left(1 + \tfrac{1}{2}\right)^2$$
$$\left(1 + \tfrac{1}{3}\right)^3$$
$$\left(1 + \tfrac{1}{4}\right)^4$$
$$\left(1 + \tfrac{1}{5}\right)^5$$

In general,

$$e^x = \lim \left(1 + \frac{x}{n}\right)^n$$

Continuous interest

- \$1000 at 35% interest after 1 year:

Continuous interest

- \$1000 at 35% interest after 1 year:

$$1000\,(1+.35) = \$1350.00$$

Continuous interest

- \$1000 at 35% interest after 1 year:

$$1000\,(1+.35) = \$1350.00$$

- Compounded once a month:

Continuous interest

- $1000 at 35% interest after 1 year:

$$1000\,(1 + .35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1 + \frac{.35}{12}\right)^{12} = \$1411.98$$

Continuous interest

- $1000 at 35% interest after 1 year:

$$1000\,(1+.35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1+\frac{.35}{12}\right)^{12} = \$1411.98$$

- Compounded once daily:

Continuous interest

- \$1000 at 35% interest after 1 year:

$$1000\,(1+.35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1+\frac{.35}{12}\right)^{12} = \$1411.98$$

- Compounded once daily:

$$1000\left(1+\frac{.35}{365}\right)^{365} = \$1418.83$$

Continuous interest

- \$1000 at 35% interest after 1 year:

$$1000\,(1 + .35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1 + \frac{.35}{12}\right)^{12} = \$1411.98$$

- Compounded once daily:

$$1000\left(1 + \frac{.35}{365}\right)^{365} = \$1418.83$$

- Compounded once a second:

Continuous interest

- $1000 at 35% interest after 1 year:

$$1000\,(1+.35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1+\frac{.35}{12}\right)^{12} = \$1411.98$$

- Compounded once daily:

$$1000\left(1+\frac{.35}{365}\right)^{365} = \$1418.83$$

- Compounded once a second:

$$1000\left(1+\frac{.35}{31536000}\right)^{31536000} = \$1419.07$$

Continuous interest

- \$1000 at 35% interest after 1 year:

$$1000\,(1 + .35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1 + \frac{.35}{12}\right)^{12} = \$1411.98$$

- Compounded once daily:

$$1000\left(1 + \frac{.35}{365}\right)^{365} = \$1418.83$$

- Compounded once a second:

$$1000\left(1 + \frac{.35}{31536000}\right)^{31536000} = \$1419.07$$

- Compounded continuously:

Continuous interest

- \$1000 at 35% interest after 1 year:

$$1000\,(1+.35) = \$1350.00$$

- Compounded once a month:

$$1000\left(1+\frac{.35}{12}\right)^{12} = \$1411.98$$

- Compounded once daily:

$$1000\left(1+\frac{.35}{365}\right)^{365} = \$1418.83$$

- Compounded once a second:

$$1000\left(1+\frac{.35}{31536000}\right)^{31536000} = \$1419.07$$

- Compounded continuously:

$$1000e^{.05} = \$1419.07$$

e to 1000 digits

2.718281828459045235360287471352662497757247093699959574966967627
7240766303535475945713821785251664274274663919320030599218174135
9662904357290033429526059563073813232862794349076323382988075319
5251019011573834187930702154089149934884167509244761460668082264
8001684774118537423454424371075390777449920695517027618386062613
3138458300075204493382656029760673711320070932870912744374704723
0696977209310141692836819025515108657463772111252389784425056953
6967707854499699679468644549059879316368892300987931277361782154
2499922957635148220826989519366803318252886939849646510582093923
9829488793320362509443117301238197068416140397019837679320683282
3764648042953118023287825098194558153017567173613320698112509961
8188159304169035159888851934580727386673858942287922849989208680
5825749279610484198444363463244968487560233624827041978623209002
1609902353043699418491463140934317381436405462531520961836908887
0701676839642437814059271456354906130310720851038375051011574770
41718986106873969655212671546889570350354...

Computed digits of e (from Wikipedia)

Number of known decimal digits of *e*

Date	Decimal digits	Computation performed by
1748	23	Leonhard Euler[19]
1853	137	William Shanks
1871	205	William Shanks
1884	346	J. Marcus Boorman
1949	2,010	John von Neumann (on the ENIAC)
1961	100,265	Daniel Shanks and John Wrench[20]
1978	116,000	Stephen Gary Wozniak (on the Apple II[21])
1994 April 1	1,000,000	Robert Nemiroff & Jerry Bonnell [22]
1999 November 21	1,250,000,000	Xavier Gourdon [23]
2000 July 16	3,221,225,472	Colin Martin & Xavier Gourdon [24]
2003 September 18	50,100,000,000	Shigeru Kondo & Xavier Gourdon [25]
2007 April 27	100,000,000,000	Shigeru Kondo & Steve Pagliarulo [26]
2009 May 6	200,000,000,000	Rajesh Bohara & Steve Pagliarulo [26]
2010 July 5	1,000,000,000,000	Shigeru Kondo & Alexander J. Yee [27]

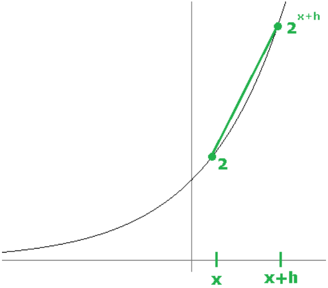
Derivatives

- e^x is best known for being its own derivative.
- It is essentially the only function with that property.
- Why?

Derivative of $y = 2^x$

Derivative is the slope of the tangent line.

Approximate it by secant lines.



$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

As $h \to 0$, we approach the slope of the tangent line.

Derivatives

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

- This is a constant times 2^x.

Derivatives

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

- This is a constant times 2^x.
- But $\frac{2^h-1}{h} \to .693$

Derivatives

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

- This is a constant times 2^x.
- But $\frac{2^h-1}{h} \to .693$
- If we try 3^x, $\frac{3^h-1}{h} \to 1.099$

Derivatives

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

- This is a constant times 2^x.
- But $\frac{2^h-1}{h} \to .693$
- If we try 3^x, $\frac{3^h-1}{h} \to 1.099$
- If we try 4^x, $\frac{4^h-1}{h} \to 1.386$ (getting worse)

Derivatives

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

- This is a constant times 2^x.
- But $\frac{2^h-1}{h} \to .693$
- If we try 3^x, $\frac{3^h-1}{h} \to 1.099$
- If we try 4^x, $\frac{4^h-1}{h} \to 1.386$ (getting worse)
- The value that works is $e \approx 2.718$: $\frac{e^h-1}{h} \to 1$

Derivatives

$$\text{slope} = \frac{2^{x+h} - 2^x}{(x+h) - x} = \frac{2^h - 1}{h} 2^x$$

- This is a constant times 2^x.
- But $\frac{2^h-1}{h} \to .693$
- If we try 3^x, $\frac{3^h-1}{h} \to 1.099$
- If we try 4^x, $\frac{4^h-1}{h} \to 1.386$ (getting worse)
- The value that works is $e \approx 2.718$: $\frac{e^h-1}{h} \to 1$

$\frac{e^h-1}{h} \approx 1$
$e^h \approx 1 + h$
$e \approx (1+h)^{1/h}$
Let $n = 1/h$
$e \approx (1 + \frac{1}{n})^n$

Logarithms

$\log_b a$ answers the question: b to what power equals a?

Logarithms

$\log_b a$ answers the question: b to what power equals a?

Example: $\log_5(25) = 2$ because $5^2 = 25$

Logarithms

$\log_b a$ answers the question: b to what power equals a?

Example: $\log_5(25) = 2$ because $5^2 = 25$

x	$y = \log_{10} x$
1	0
10	1
100	2
1000	3
10000	4

A multiplicative change in x corresponds to an additive change in y.

Formally,

$$\log(ab) = \log(a) + \log(b)$$

Logarithms and Integrals

$\int x^n \, dx = \frac{x^{n+1}}{n+1}$. But what if $n = -1$?

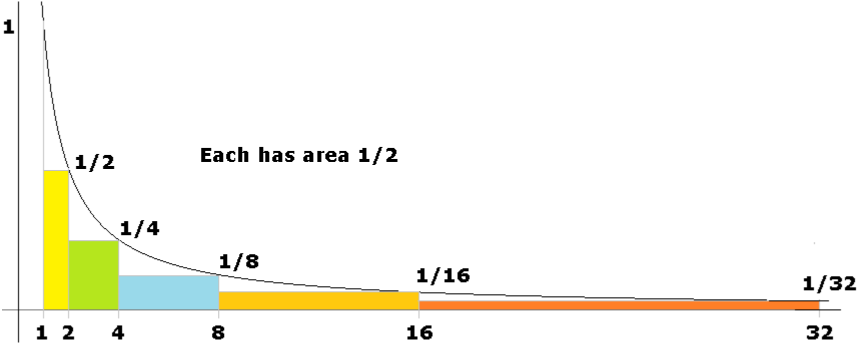
Logarithms and Integrals

$\int x^n \, dx = \frac{x^{n+1}}{n+1}$. But what if $n = -1$?

$\int_1^x \frac{1}{t} \, dt$ is the area under $y = 1/x$ from $t = 1$ to $t = x$.

$\int x^n \, dx = \frac{x^{n+1}}{n+1}$. But what if $n = -1$?

$\int_1^x \frac{1}{t} \, dt$ is the area under $y = 1/x$ from $t = 1$ to $t = x$.



A multiplicative change in x corresponds to an additive change in the area.

What base?

This leads to

$$\int_1^x \frac{1}{t}\,dt = \log x.$$

But what is the base?

What base?

This leads to

$$\int_1^x \frac{1}{t}\,dt = \log x.$$

But what is the base?

The base is e.

$$\int_1^x \frac{1}{t}\,dt = \log_e x = \ln x.$$

What base?

This leads to

$$\int_1^x \frac{1}{t}\,dt = \log x.$$

But what is the base?

The base is e.

$$\int_1^x \frac{1}{t}\,dt = \log_e x = \ln x.$$

But why not something else, like base 7 or base 443.18?

What base?

This leads to

$$\int_1^x \frac{1}{t}\,dt = \log x.$$

But what is the base?

The base is e.

$$\int_1^x \frac{1}{t}\,dt = \log_e x = \ln x.$$

But why not something else, like base 7 or base 443.18?

Why base e

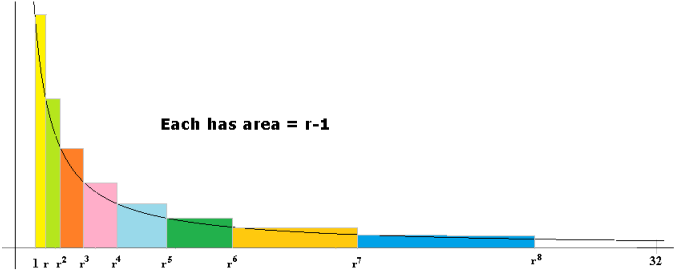
Say we want $\displaystyle\int_1^{32} \frac{1}{x}\,dx$.

Why base e

Say we want $\displaystyle\int_1^{32} \frac{1}{x}\, dx$.

Suppose instead of powers of 2, we use something smaller, like powers of $r = 1.5$

The smaller rectangles will fit the area more closely.



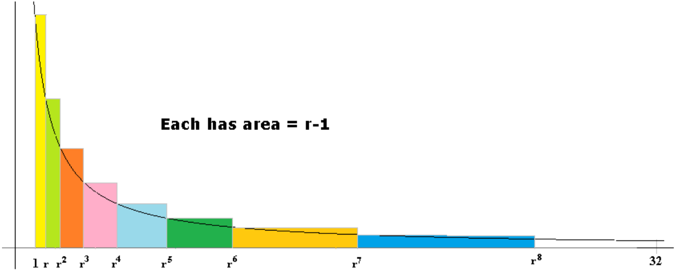
How many rectangles will there be?

Why base e

Say we want $\displaystyle\int_1^{32} \frac{1}{x}\,dx$.

Suppose instead of powers of 2, we use something smaller, like powers of $r = 1.5$

The smaller rectangles will fit the area more closely.



How many rectangles will there be?

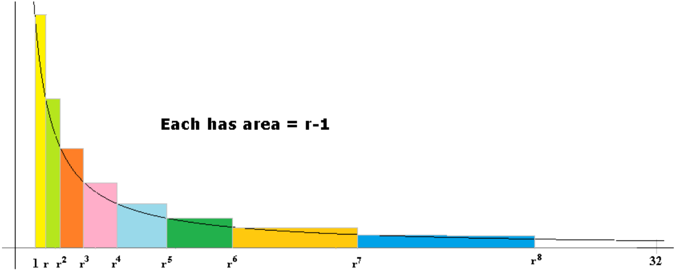
Answer: Find the largest power of r less than 32.

Why base e

Say we want $\displaystyle\int_1^{32} \frac{1}{x}\,dx$.

Suppose instead of powers of 2, we use something smaller, like powers of $r = 1.5$

The smaller rectangles will fit the area more closely.



How many rectangles will there be?

Answer: Find the largest power of r less than 32.

In other words, solve $r^x = 32$. We get $x = \frac{\log(32)}{\log r}$.

Why base e, cont.

The area is then $\frac{\log(32)}{\log r}(r-1)$.

Why base e, cont.

The area is then $\frac{\log(32)}{\log r}(r-1)$.

Suppose we go smaller than 1.5, say to $r = 1 + \frac{1}{n}$ for some small value of n.

Why base e, cont.

The area is then $\frac{\log(32)}{\log r}(r-1)$.

Suppose we go smaller than 1.5, say to $r = 1 + \frac{1}{n}$ for some small value of n.

The area is then

$$\begin{aligned}
&\frac{\log(32)}{\log(1+\frac{1}{n})}(1+\frac{1}{n}-1) \\
=&\frac{\log(32)}{n\log(1+\frac{1}{n})} \\
=&\frac{\log(32)}{\log(1+\frac{1}{n})^n} \\
=&\log_{(1+\frac{1}{n})^n}(32)
\end{aligned}$$

As $n \to \infty$, this becomes $\log_e(32)$.

e in calculus

In summary, e is so important in calculus because:

- $f(x) = e^x$ is (more or less) the only function whose derivative is itself
- $\log_e(x)$ is the antiderivative of $\frac{1}{x}$.

Infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From Taylor series or binomial theorem on $(1 + \frac{1}{n})^n$

Infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From Taylor series or binomial theorem on $(1 + \frac{1}{n})^n$

$1 + \frac{1}{1!} = 2$

$1 + \frac{1}{1!} + \frac{1}{2!} = 2.5$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.666667$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708333$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.716666$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718055$

Infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

From Taylor series or binomial theorem on $(1 + \frac{1}{n})^n$

$1 + \frac{1}{1!} = 2$

$1 + \frac{1}{1!} + \frac{1}{2!} = 2.5$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.666667$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708333$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.716666$

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718055$

In general, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Euler's formula

e^x is directly related to $\sin x$ and $\cos x$:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's formula

e^x is directly related to $\sin x$ and $\cos x$:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

If you plug in $\theta = \pi$, you get...

Euler's formula

e^x is directly related to $\sin x$ and $\cos x$:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

If you plug in $\theta = \pi$, you get...

"The most remarkable formula in mathematics"

$$e^{i\pi} + 1 = 0$$

e is irrational

e is irrational

Proof: Suppose $e = \frac{p}{q}$. Using the power series for e, we have

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!} + \frac{1}{(q+1)!} + \dots$$

e is irrational

Proof: Suppose $e = \frac{p}{q}$. Using the power series for e, we have

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!} + \frac{1}{(q+1)!} + \ldots$$

Multiply both sides by $q!$ to get

$$p(q-1)! = q! + q! + q(q-1) \ldots 3 \cdot 2 + \cdots + 1 + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots$$

e is irrational

Proof: Suppose $e = \frac{p}{q}$. Using the power series for e, we have

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!} + \frac{1}{(q+1)!} + \ldots$$

Multiply both sides by $q!$ to get

$$p(q-1)! = q! + q! + q(q-1)\ldots 3 \cdot 2 + \cdots + 1 + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots$$

The left side is an integer. The right side is not because $\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots < 1$.

Contradiction!

If you want the details. . .

Even if $q = 2$, we have $\frac{1}{3} + \frac{1}{12} + \frac{1}{60} + \ldots$, which is small

If you want the details...

Even if $q = 2$, we have $\frac{1}{3} + \frac{1}{12} + \frac{1}{60} + \dots$, which is small

Formally,

$$
\begin{aligned}
&\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \dots \\
\leq &\frac{1}{2+1} + \frac{1}{(2+1)(2+2)} + \dots \\
< &\frac{1}{3} + \frac{1}{3^2} + \dots \\
= &\frac{1}{1-\frac{1}{3}} - 1 \\
= &\frac{1}{2}
\end{aligned}
$$

Catenary

What is the shape of a wire hanging between two points?

Catenary

What is the shape of a wire hanging between two points?

It's called a *catenary* and its equation is $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$.

A famous Catenary

(Gateway Arch in St. Louis)

Derangements

How many ways are there to rearrange the numbers 1, 2, . . . , n?

Derangements

How many ways are there to rearrange the numbers 1, 2, . . . , n?

$n = 2$: 12, 21 (2 total)

$n = 3$: 123, 132, 213, 231, 312, 321 (6 total)

$n = 4$: 1234, 1243, . . . , 4321 (24 total)

Derangements

How many ways are there to rearrange the numbers 1, 2, . . . , n?

$n = 2$: 12, 21 (2 total)

$n = 3$: 123, 132, 213, 231, 312, 321 (6 total)

$n = 4$: 1234, 1243, . . . , 4321 (24 total)

In general it's $n!$.

Derangements

How many ways are there to rearrange the numbers 1, 2, . . . , n?

$n = 2$: 12, 21 (2 total)

$n = 3$: 123, 132, 213, 231, 312, 321 (6 total)

$n = 4$: 1234, 1243, . . . , 4321 (24 total)

In general it's $n!$.

How many ways are there to rearrange so that no number stays fixed?

Derangements

How many ways are there to rearrange the numbers 1, 2, . . . , n?

$n = 2$: 12, 21 (2 total)

$n = 3$: 123, 132, 213, 231, 312, 321 (6 total)

$n = 4$: 1234, 1243, . . . , 4321 (24 total)

In general it's $n!$.

How many ways are there to rearrange so that no number stays fixed?

This is called a *derangement*.

A curious number

n	Derangements (d_n)	Rearrangements (r_n)	d_n/r_n
1	0	1	0.000000
2	1	2	0.500000
3	2	6	0.333333
4	9	24	0.375000
5	44	120	0.366667
6	265	720	0.368155
7	1854	5040	0.367857
8	14833	40320	0.367881
9	133496	362880	0.367879
10	1334961	3628800	0.367879

What value is the d_n/r_n approaching?

A curious number

n	Derangements (d_n)	Rearrangements (r_n)	d_n/r_n
1	0	1	0.000000
2	1	2	0.500000
3	2	6	0.333333
4	9	24	0.375000
5	44	120	0.366667
6	265	720	0.368155
7	1854	5040	0.367857
8	14833	40320	0.367881
9	133496	362880	0.367879
10	1334961	3628800	0.367879

What value is the d_n/r_n approaching?

$$\frac{1}{e}$$

- e is called *Euler's constant* or *Napier's constant*

History

- e is called *Euler's constant* or *Napier's constant*
- In early 1600s John Napier invented logarithms

History

- e is called *Euler's constant* or *Napier's constant*
- In early 1600s John Napier invented logarithms
- The method he used to generate his tables involved e although he didn't recognize it in the way we now know it

History

- e is called *Euler's constant* or *Napier's constant*
- In early 1600s John Napier invented logarithms
- The method he used to generate his tables involved e although he didn't recognize it in the way we now know it
- Jacob Bernoulli discovered e in late 1600s studying compound interest

History

- e is called *Euler's constant* or *Napier's constant*
- In early 1600s John Napier invented logarithms
- The method he used to generate his tables involved e although he didn't recognize it in the way we now know it
- Jacob Bernoulli discovered e in late 1600s studying compound interest
- Leonhard Euler in 1700s gave it the symbol e

History

- e is called *Euler's constant* or *Napier's constant*
- In early 1600s John Napier invented logarithms
- The method he used to generate his tables involved e although he didn't recognize it in the way we now know it
- Jacob Bernoulli discovered e in late 1600s studying compound interest
- Leonhard Euler in 1700s gave it the symbol e
- Euler did a lot with e

- Bell curve $y = e^{-x^2/2}$

Interesting facts and formulas

- Bell curve $y = e^{-x^2/2}$
- $\int_0^\infty e^{-x^2/2} = \sqrt{\pi/2}$

Interesting facts and formulas

- Bell curve $y = e^{-x^2/2}$
- $\int_0^\infty e^{-x^2/2} = \sqrt{\pi/2}$
- Infinite product: $2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdot \frac{e^{1/5}}{e^{1/6}} \cdot \frac{e^{1/7}}{e^{1/8}} \dots$

Interesting facts and formulas

- Bell curve $y = e^{-x^2/2}$
- $\int_0^\infty e^{-x^2/2} = \sqrt{\pi/2}$
- Infinite product: $2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdot \frac{e^{1/5}}{e^{1/6}} \cdot \frac{e^{1/7}}{e^{1/8}} \ldots$
- A continued fraction:

$$e = 2 + \cfrac{1}{1 + \frac{1}{2+\frac{2}{3+\frac{3}{4+\frac{4}{5+\ldots}}}}}$$

Interesting facts and formulas

- Bell curve $y = e^{-x^2/2}$
- $\int_0^\infty e^{-x^2/2} = \sqrt{\pi/2}$
- Infinite product: $2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdot \frac{e^{1/5}}{e^{1/6}} \cdot \frac{e^{1/7}}{e^{1/8}} \cdots$
- A continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ldots}}}}}$$

- Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Interesting facts and formulas

- Bell curve $y = e^{-x^2/2}$
- $\int_0^\infty e^{-x^2/2} = \sqrt{\pi/2}$
- Infinite product: $2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdot \frac{e^{1/5}}{e^{1/6}} \cdot \frac{e^{1/7}}{e^{1/8}} \cdots$
- A continued fraction:

$$e = 2 + \cfrac{1}{1 + \frac{1}{2+\frac{2}{3+\frac{3}{4+\frac{4}{5+\dots}}}}}$$

- Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
- $\sqrt{e^\pi} = \sqrt[i]{i}$

Interesting facts and formulas

- Bell curve $y = e^{-x^2/2}$
- $\int_0^\infty e^{-x^2/2} = \sqrt{\pi/2}$
- Infinite product: $2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdot \frac{e^{1/5}}{e^{1/6}} \cdot \frac{e^{1/7}}{e^{1/8}} \cdots$
- A continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \dots}}}}}$$

- Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
- $\sqrt{e^\pi} = \sqrt[i]{i}$
- Google's 2004 IPO announced they were trying to raise \$2,718,281,828.

Thanks!

Thank you for your attention.