Smalltalk: Newton's Method, Chaos, and Fractals

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February 2, 2017

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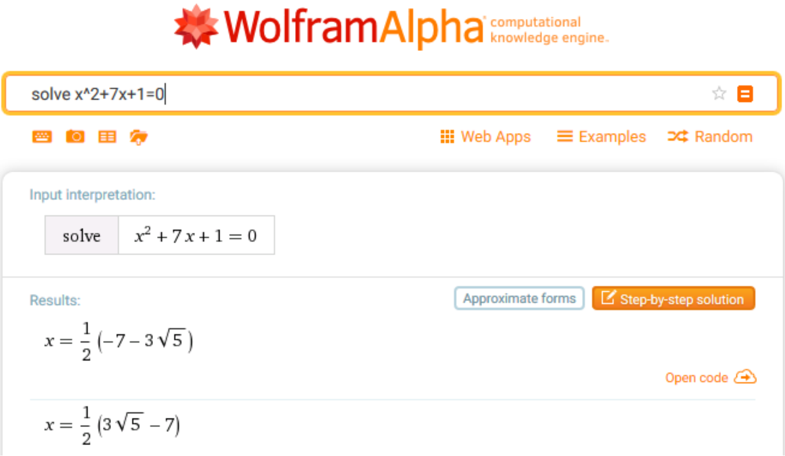
Simplifies to $\frac{1}{2}(-7 + \sqrt{45})$ and $\frac{1}{2}(-7 - \sqrt{45})$

Let's Check Our Work. . .



A Harder Question

Solve $x^5 + 7x + 1 = 0$.

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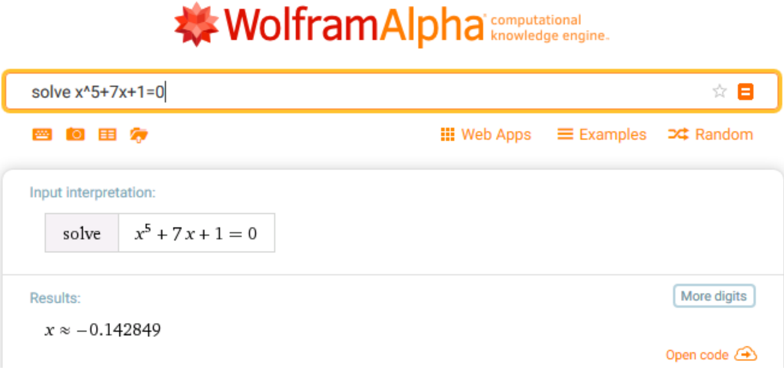
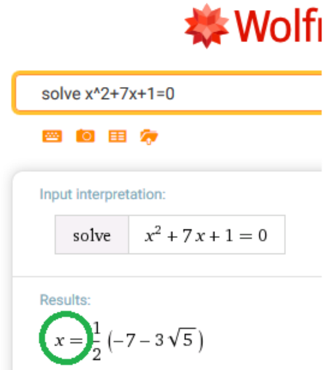
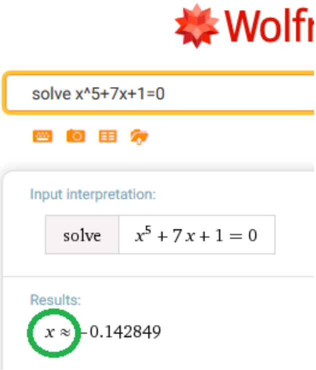
Answer? Hmmm...

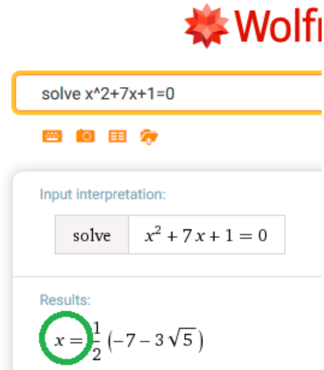
Let's Try Wolfram. . .

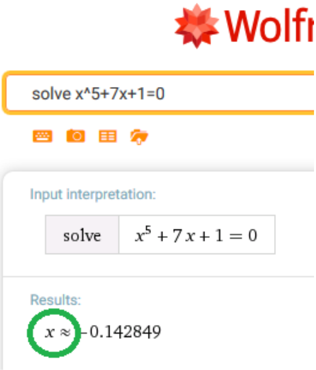


Look Carefully









Why $=$ in the first and $\approx$ in the second?

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- And this includes integrals and differential equations

- So what do we do?

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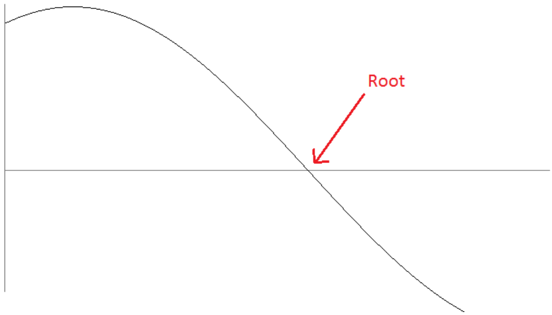
- It's a series of steps to follow. Each step hopefully gets us a little closer to the exact answer.
- Newton's Method is one such numerical method.

Solving Equations

Solving an equation like $x^5 + 7x + 1 = 0$ is looking for where the graph crosses the x-axis.



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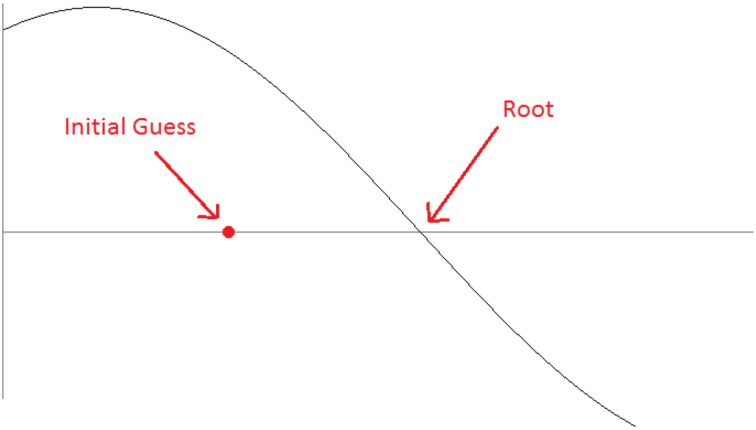
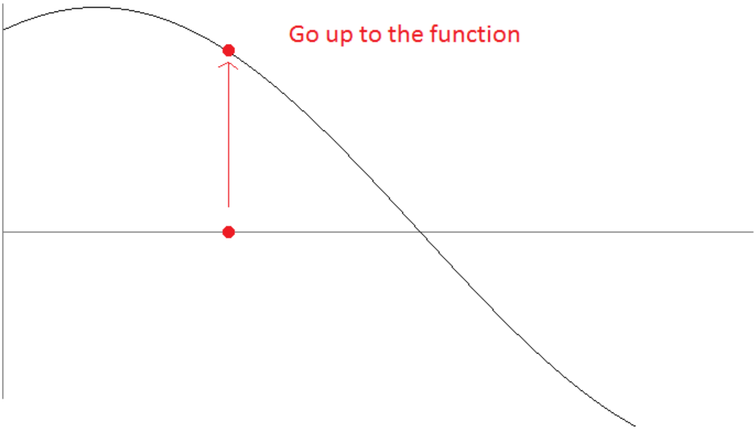
- Then follow the following sequence of steps.

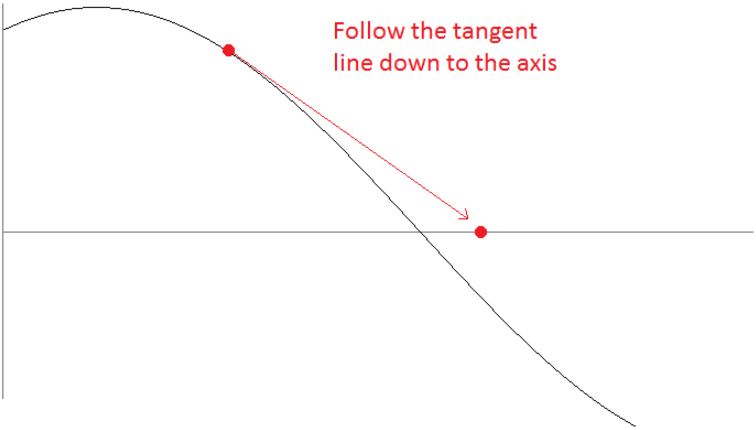
Step 1



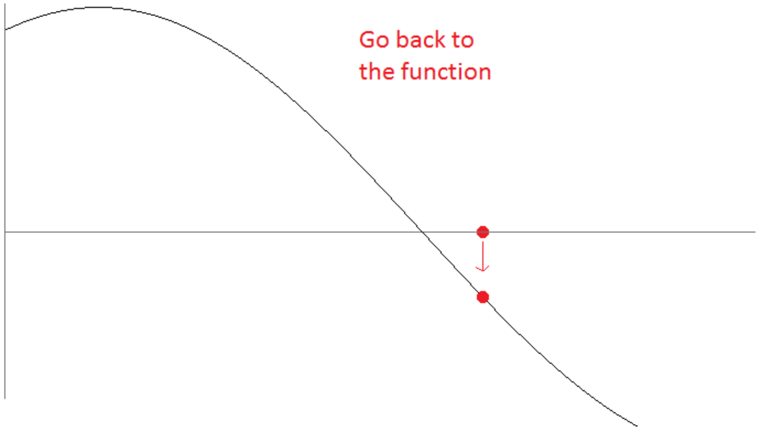
Step 2



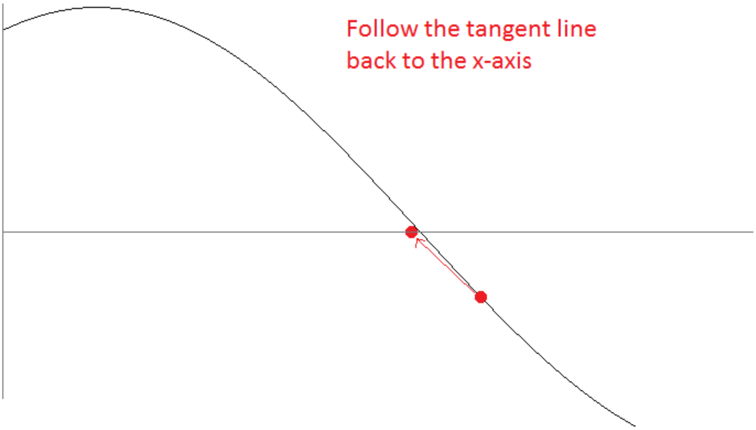
Step 3



Step 4



Step 5



Steps 6, 7. . .

Repeat these steps a few more times.

The Overall Process

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2. Keep repeating the following until we get tired:
   1. Go to the function
   2. Follow the tangent line to the axis

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$$\text{improved guess} = \text{old guess} - \frac{f(\text{old guess})}{f'(\text{old guess})}$$

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Or, in more common notation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_0 = \text{initial guess}$$

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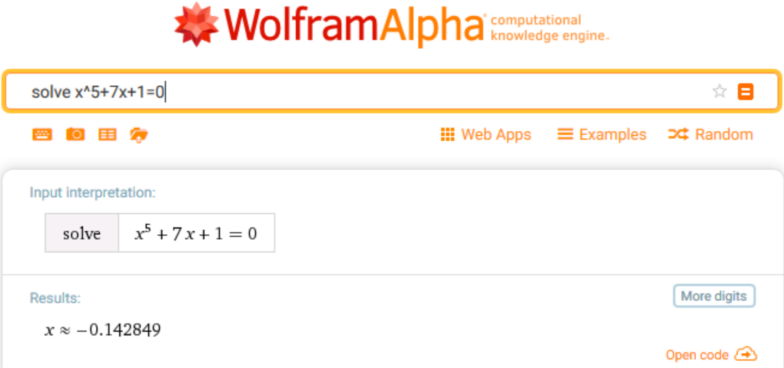
- $x_2 = .25 - \dfrac{.25^5 + 7 \cdot .25 + 1}{5 \cdot .25^4 + 7} = -0.1419\ldots$
- $x_3 = -.1419 \cdots - \dfrac{(-0.1419\ldots)^5 + 7 \cdot (-0.1419\ldots) + 1}{5 \cdot (-0.1419\ldots)^4 + 7} = -0.1428\ldots$

Look familiar?



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- The first 36 values, using a starting value of .4:

  `.960, .154, .520, .998, .006, .025, .099, .358, .919, .298, .837, .547,`
  `.991, .035, .135, .466, .995, .018, .071, .263, .774, .699, .842, .532,`
  `.996, .016, .064, .241, .732, .785, .676, .876, .434, .983, .068, .252,`
  `.754, .742, .765, .719, .808, .620, .942, .219, .683, .866, .464, .995`

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- Looks pretty random...

Even Stranger...

Iterations for two very close starting values:

.400	.401
.960	.960
.154	.151
.520	.512
.998	.999
.006	.002
.025	.009
.099	.036
.358	.137
.919	.474

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.919	.919
.298	.299
.837	.838
.547	.543
.991	.993
.035	.029
.135	.113
.466	.400
.995	.960
.018	.153
.071	.519

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This is an example of *chaos*.

- Irregular, but not totally random
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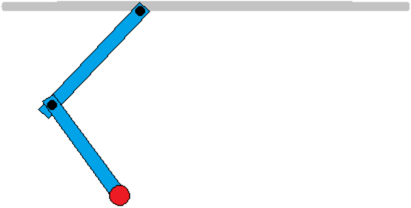
- Even if our starting values were vanishingly close, say only 10^{-20} apart, it would only take several dozen iterations for them to start to diverge.

The Double Pendulum: Another Chaotic System



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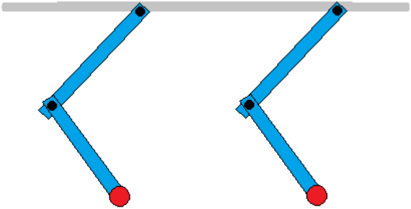

A double pendulum is a pendulum attached to another pendulum.

If we change the starting angle by something as small as just the width of an atom (like a .0000000001 difference), after 30-60 seconds the pendulum will be doing something completely different.

Another Chaotic System: The Magnetic Pendulum

No matter how hard you try to start it in the same location, each time you release it, after a few seconds, it will be doing something different.

Another Chaotic System: Plinko

Another Chaotic System: The Weather

- The Butterfly Effect — A butterfly flapping its wings in Japan can mean the difference between a tornado and a sunny day six months later in Texas

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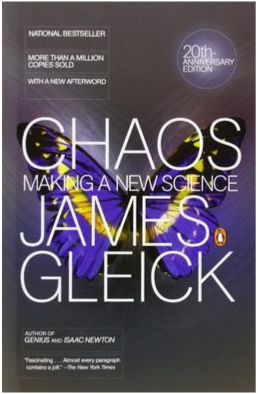
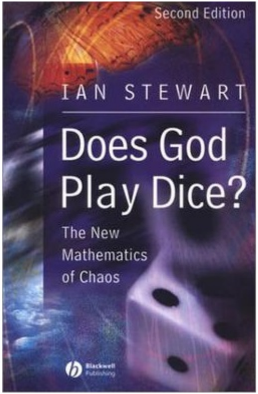
- Weather is a chaotic system; this is why we can't predict the weather more than a few days out.

Another Chaotic System: Life

Before leaving for school, I stop to look at myself in the mirror for a few seconds. When I get to the intersection at the road, there is now a car coming that I wouldn't have met. While waiting, I see a quarter next to my bike and pick it up. If I hadn't picked it up, a few days later someone walking by would have thrown out their back while trying to pick it up. Later that day at the doctor's office that person met someone else. She was moving to Houston the next day, but she and the guy that threw out his back have a good conversation and decide to stay in touch. Later they start dating, get married. That marriage would not have happened if I hadn't stopped to look in the mirror.

Some Good Books on Chaos





Chaos: Making a New Science, James Gleick, 1987.

Does God Play Dice? Ian Stewart, 1989.

Solutions of $x^3 - 1 = 0$?

Imaginary Numbers

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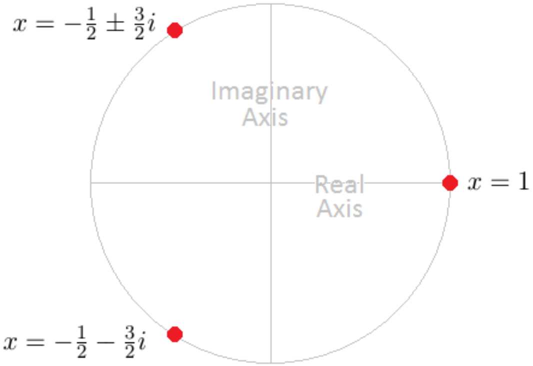

Newton's Method Works with Imaginary Numbers

```
>>> x = 1.8+.9j
>>> for i in range(5):
        x = x - (x**3-1)/(3*x**2)
        print(x)


(1.2493827160493827+0.534156378600823j)
(0.9576629335418402+0.22558414417185063j)
(0.9465871950754545-0.003310420182178475j)
(1.0030569479535545+0.00039500570537458O7j)
(1.0000091528504522+2.40041437750974e-06j)
```

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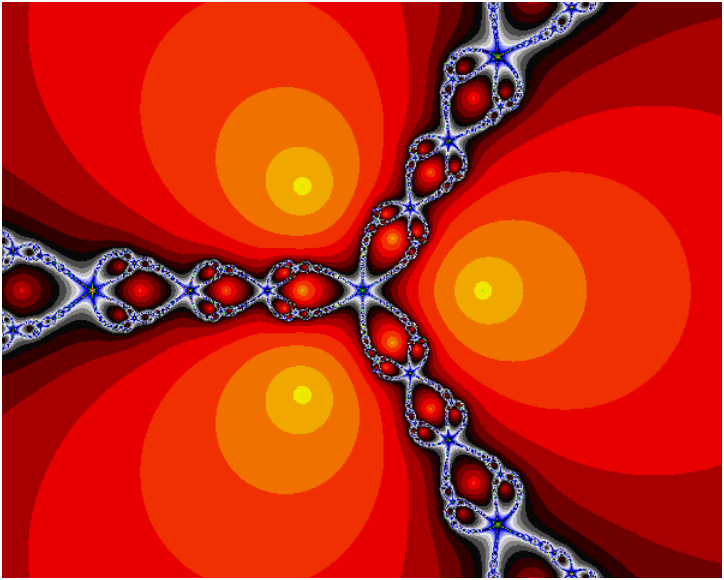
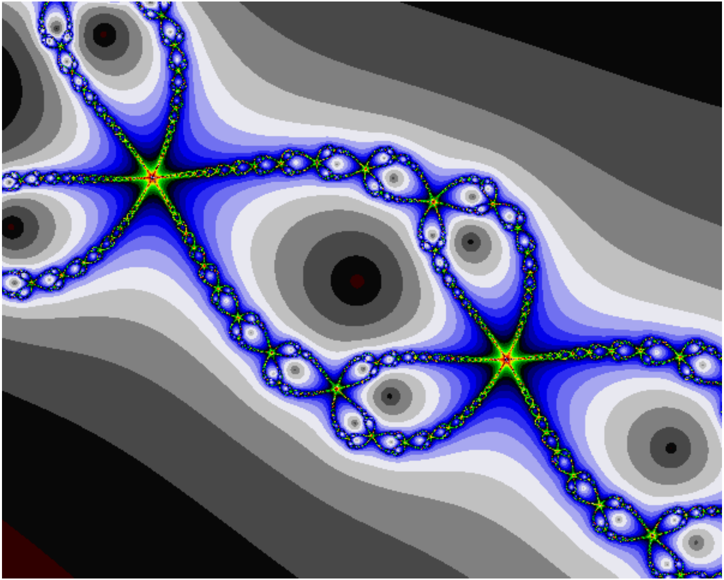
- Some will go to one root, some to other.
- The closer they are to a root, the more quickly they will get there.
- Try each starting value in the range from say -2 to 2 in the real and imaginary directions.
- Color each one based on how long it takes until it is within .00001 of a root.

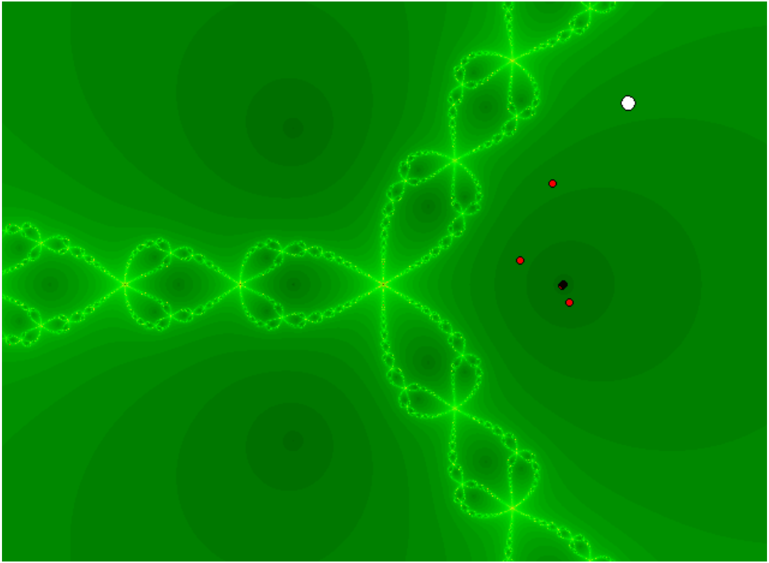
This Is What We Get



A Zoomed-In Version



Example Orbits of Iterations



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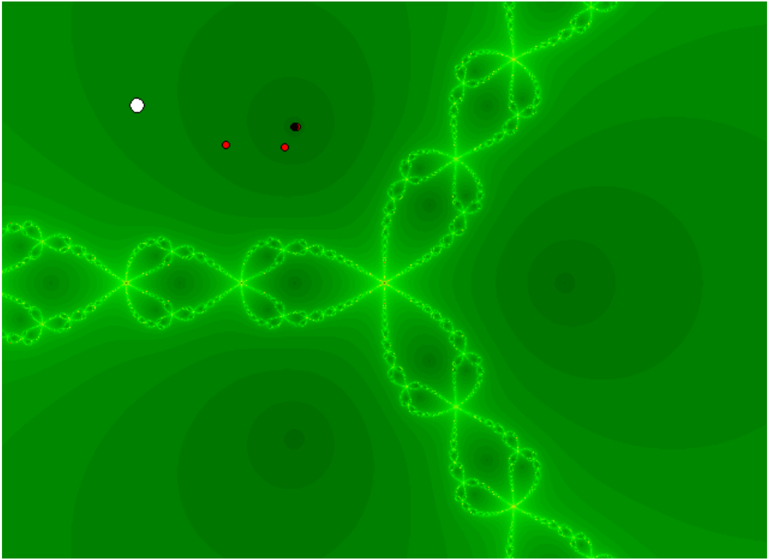

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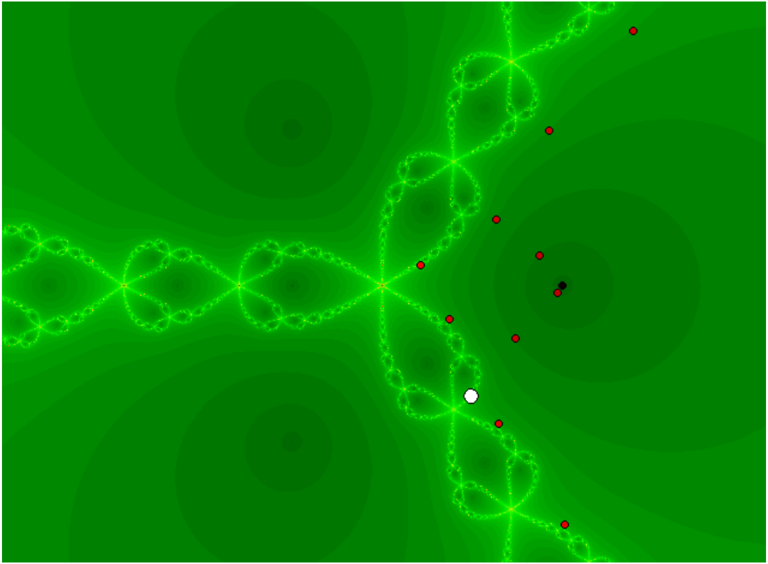

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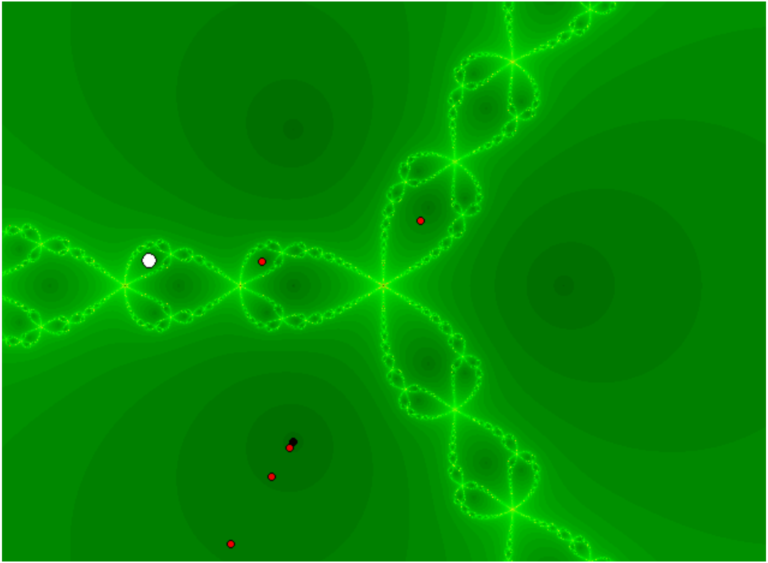
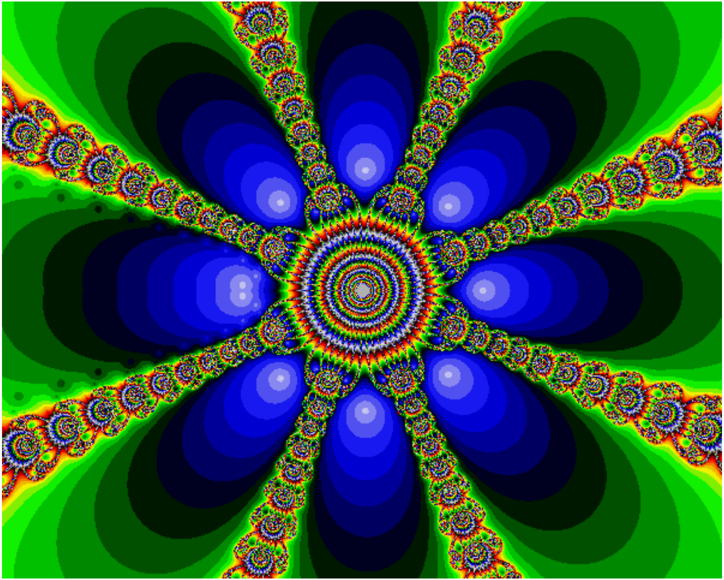


Newton's Method on $z^9 - 1 = 0$ (Roughly)



Newton's Method on $z^c - 1 = 0$ (c is imaginary)

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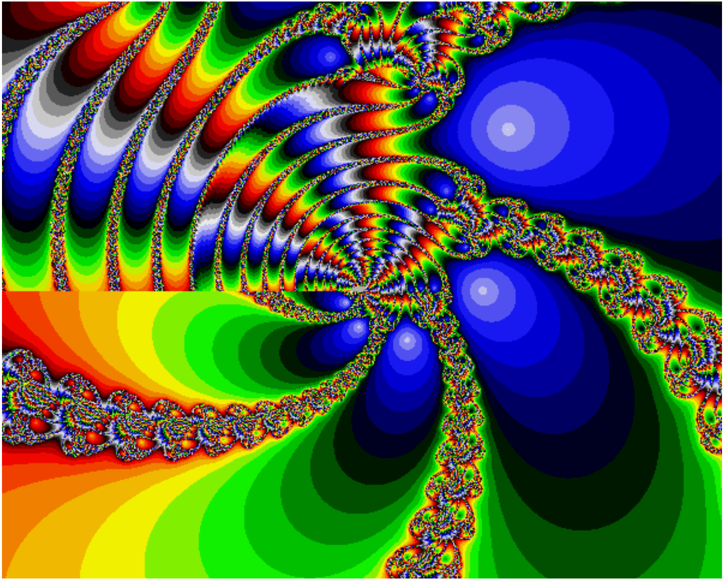
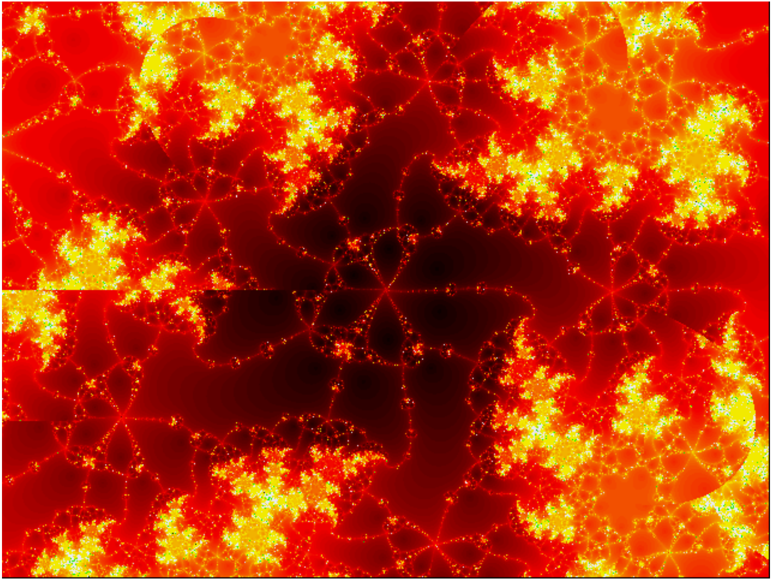
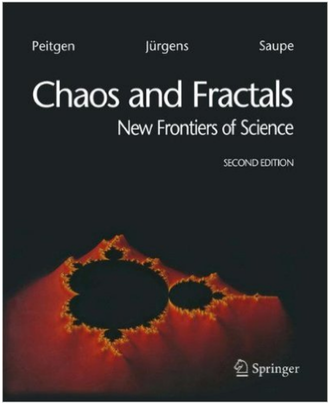


Netwon's Method on $z^{-4.7+.3i} + z - 1$



Good Book on Fractals



Chaos and Fractals: New Frontiers of Science, 2nd ed.,
Heinz-Otto Peitgen, Hartmut Jürgens, & Dietmar Saupe, 2004

Thanks for your attention!