Questions of Prime Importance

Brian Heinold

Primes

A *prime number* is an integer greater than 1 whose only divisors are 1 and itself.

The first few primes:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, ...

Number of primes

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Proportion of integers between 1 and n that are prime $\approx \dfrac{1}{\ln n}$.

Roughly $\dfrac{1}{\ln(1000)} = 14.5\%$ between 2 and 1000

Roughly $\dfrac{1}{\ln(1000000)} = 7.2\%$ between 2 and 1,000,000

Number of primes, continued

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Better estimate: Number of primes less than n is approximately

$$\int_2^n \frac{1}{\ln x}\, dx$$

It predicts 78628 (versus exact value 78498).

Mersenne primes

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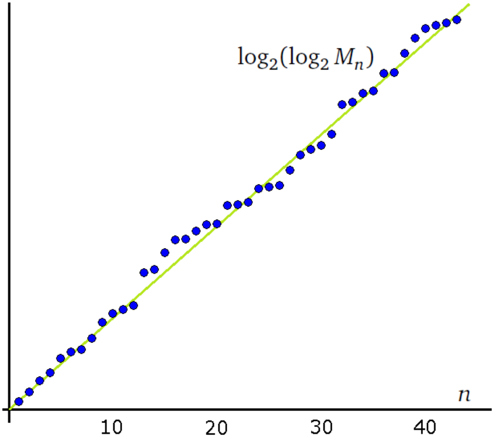
The exponents n in $2^n - 1$ corresponding to Mersenne primes are 2, 3, 5, 7, 13, 17, 19, 31, 61, ... Plotting the logarithm of these numbers gives:

Log of the exponent of Mersenne primes



Sophie Germain primes

Sophie Germain primes are primes p where $2p + 1$ is also prime.

2 (because $2 \cdot 2 + 1 = 5$ is also prime
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But no one knows for sure. These numbers get very big, very fast ($2^{2^{32}} + 1$ has about a billion digits)

Patterns

Red curve: $n^2 + n + 41$ (generates primes at a high rate)

Progressions

Are there infinitely many primes that end in 01?

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Are there infinitely many primes that end in 123?
1123, 2123, 3123, 4123, 5123, 6123, 7123, **8123**, 9123, 10123, ...

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Numbers ending in 01: $a = 100$, $b = 1$
Numbers ending in 123: $a = 1000, b = 123$

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Numbers ending in 123: $a = 1000, b = 123$

Applies to many other cases as well.

Progressions, continued

How about sequences of equally-spaced primes?

3, 5, 7 (each 2 apart)
5, 11, 17, 23, 29 (each 6 apart)

Is it possible to find arbitrarily long progressions?

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Yes [Green & Tao, 2004]

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Longest known is 25 terms long, starting at 43,142,846,595,714,191

A simple question

Is there always a prime between consecutive squares?

between 1 and 4: 2, 3
between 4 and 9: 5, 7
between 9 and 16: 11, 13
between 16 and 25: 17, 19, 23

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between 10,000 and 10,201: 23 primes
between 1,000,000 and 1,002,001: 152 primes

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No one can prove the fact, though it is very likely true.

A simple question, continued

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Gaps between first 100 primes:

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Twin Primes

Twin primes are primes that are 2 apart.

..., 23, 29, 31, 37, **41**, **43**, 47, 53, **59**, **61**, 67, **71**, **73**, 79, 83, ...

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- Recent work (Zhang et al. 2013-2014): There are infinitely many primes p such that one of $p+2, p+4, \ldots, p+246$ is also prime.
- Largest known twin prime pair: $3756801695685 \cdot 2^{666669} \pm 1$ (about 200,000 digits)

Goldbach's Conjecture

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$6 = 3 + 3$
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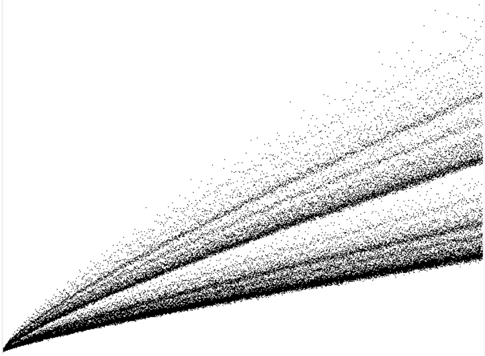
$100 = 3 + 97,\ 11 + 89,\ 17 + 83,\ 29 + 71,\ 41 + 59,\ 47 + 53$
28 ways to write 1000
127 ways to write 10,000
810 ways to write 100,000

Goldbach's Conjecture, continued

Here is a graph showing how the number of possible ways to write a number as a sum of two primes increases with n. (The horizontal axis runs from $n = 4$ to $n =$100,000 and the vertical axis runs to about 2000.)



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Partial results:

- Chen Jingrun early 1970s: Every sufficiently large even number can be written as a sum $p + q$, where p is prime and q is either prime or a product of two primes.
- *Weak Goldbach conjecture* Every odd number greater than 7 is the sum of three primes. Seems to have been proved true in 2013.
- The (strong) Goldbach conjecture has been verified by computer for all integers less than 10^{18}.

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- If RH is true, it would mean that we have a pretty good understanding of distribution of primes, that they are distributed pretty regularly.
- If false, then we don't understand primes as well as we thought.

The *zeta function* (very important in number theory):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad \text{(diverges)}$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.202$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Zeta function and the Riemann Hypothesis

The Zeta Function has a connection with primes:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

For example:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \cdots .$$

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The locations of those zeroes have important consequences for what we know about primes.

Thanks!

Thank you for your attention.