Smalltalk: The Poincaré Conjecture

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Statement of the problem, as given on Wikipedia:

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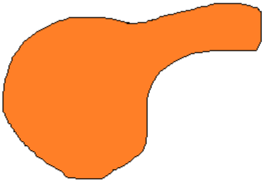
Let's examine each term so we can understand the whole statement.

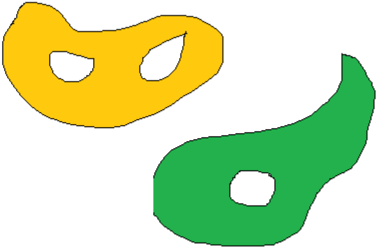
Simply Connected

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Roughly speaking, a 2D region is simply connected if it has no holes, like below.



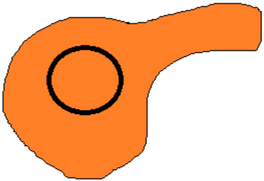
Simply Connected



Not Simply Connected

Simply Connected

More formally, a region is simply connected if every closed curve can be shrunk to a point without leaving the region.



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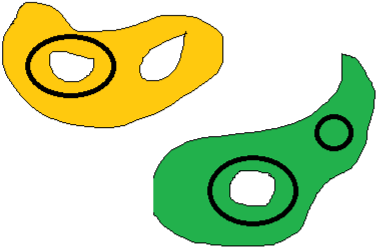

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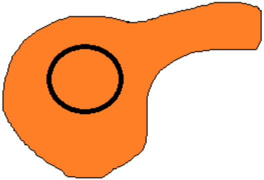

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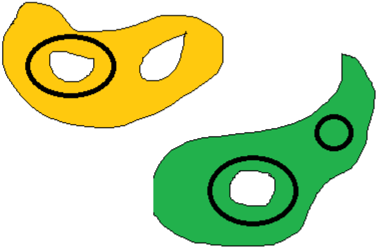
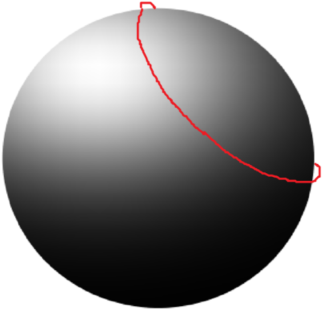
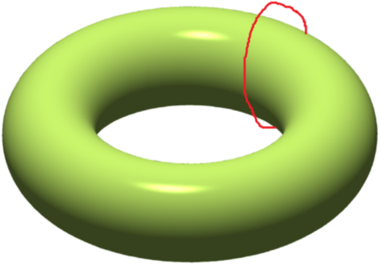


Not Simply Connected

Imagine hitting the circles with a shrink ray. Will they be able to get as small as we like, or will the region get in the way?

This applies in higher dimensions as well. For 3D shapes, there can be holes, but they can't "go all the way through" the object. The same curve definition applies here as well.



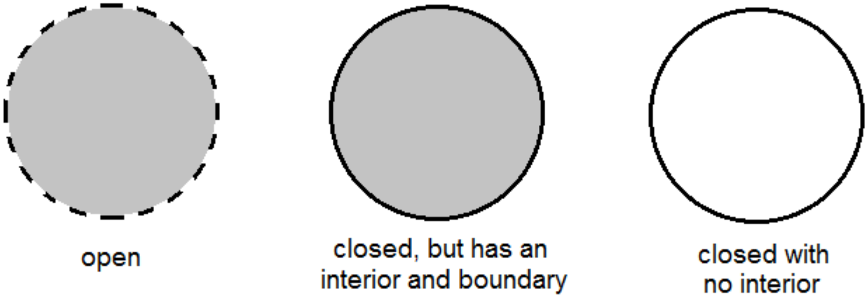


Closed

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

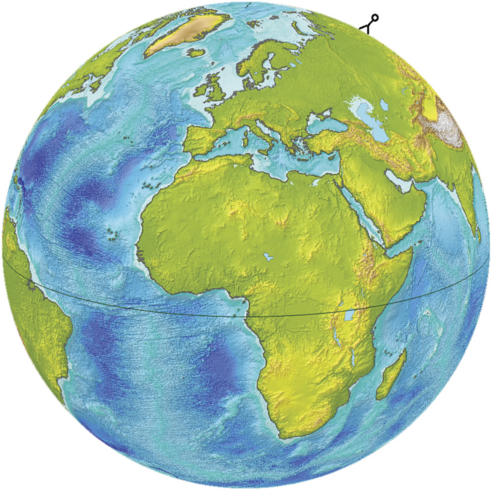
Closed

The rightmost figure below is the type that is relevant to the Poincaré Conjecture.



Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Ever think about how the earth is round, but it sure seems flat?



That's roughly the definition of a manifold. It's a shape where if you're anywhere on it, and are small enough in relation to the shape, you wouldn't be able to tell it from a flat region.

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Someone looking at it from the outside could tell it wasn't flat, but it certainly would appear flat to you no matter where you are on the surface.

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Manifolds

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A small enough creature living anywhere on the manifolds would not be able to distinguish them from perfectly flat lines.

2-Manifolds

Here are some 2D manifolds. The sphere on the right consists only of the outside surface.

3-Manifolds and Higher dimensions

A 3-Manifold adds a third dimension. The space we live in is a 3-manifold.

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There's nothing that stops us mathematically from defining 4-manifolds, 5-manifolds, etc., but they do become difficult to visualize.

Homeomorphic

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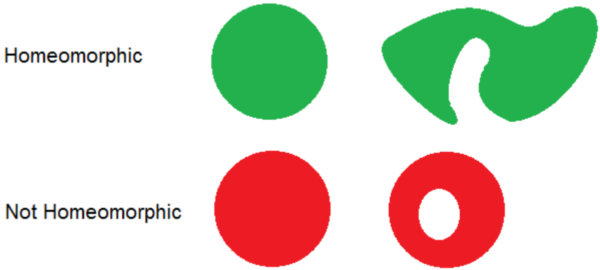

Homeomorphic

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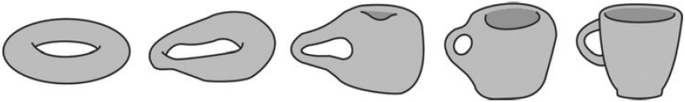

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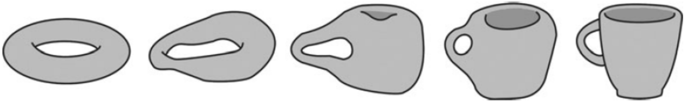

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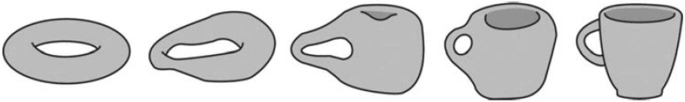


The idea is that a coffee cup and a doughnut are homeomorphic.

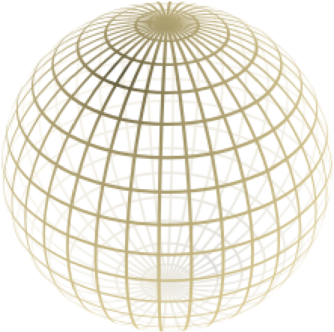
Note, however, that neither is homeomorphic to a ball because the donut and coffee cup both have holes, but a ball doesn't.

3-sphere

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Spheres

What we typically call a sphere is a 2-dimensional object. It's the surface of a perfectly round ball.

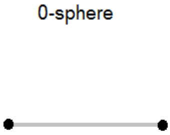


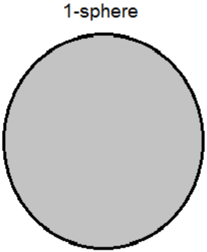
Spheres

A 0-dimensional sphere is two points, the boundary of an interval.

A 1-dimensional sphere is a circle, the boundary of a disk.

A 2-dimensional sphere is a the boundary of a ball.

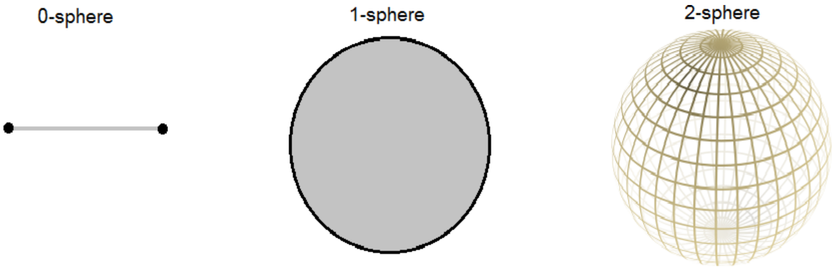




Spheres

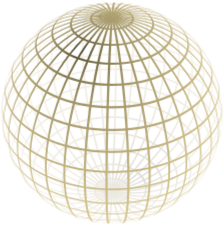
Another way to define a circle is as all the points in the plane that are equidistant from a given center point.

You can define a 2D sphere as all the points in space that are equidistant from a center point.



3-Sphere

Even though a sphere is a 2D object, we really need 3 dimensions to truly appreciate it.



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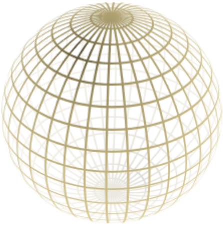

Imagine you are a flat 2D person living on a piece of paper. What would a sphere look like to you?

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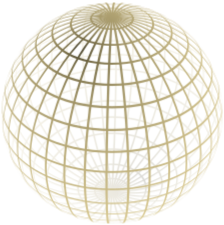
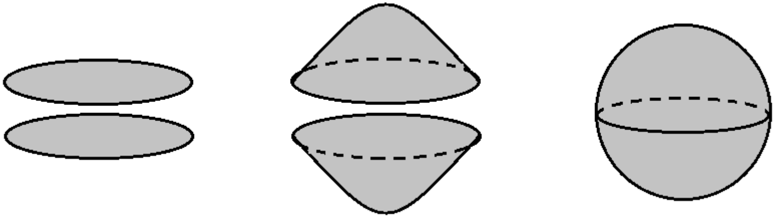


Imagine you are a flat 2D person living on a piece of paper. What would a sphere look like to you?

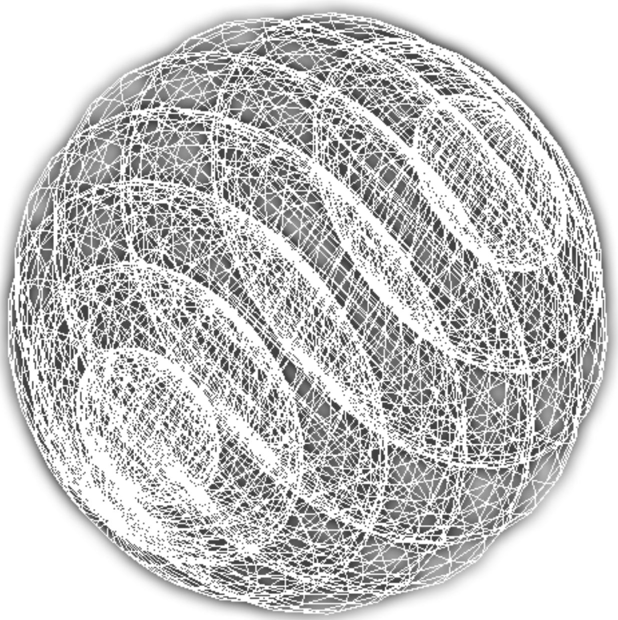
Similarly, even though a 3-sphere is a 3D object and we live in 3D, we would really need 4 dimensions to appreciate it.

One way to create a sphere is to take two disks, glue them along their boundary circle, and stretch them out until they have the shape of hemispheres.



To create a 3-sphere, one could take two balls, glue them along their boundary spheres, and stretch them. You would need to be in 4D to do this.

3-Sphere



Henri Poincaré

Sometimes called the last true polymath, the last person to be able to work in all of the branches of mathematics of his day.

The Poincaré Conjecture

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

So, roughly, it's saying that any 3D shape with no holes that appears "flat" to small creatures living on it can be continuously stretched and bent until it becomes a 3-sphere.

Here is Henri Poincaré's original 1904 statement of the problem:

Est-il possible que le groupe fondamental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?

By the 1930s it had become one of the most famous problems in math.

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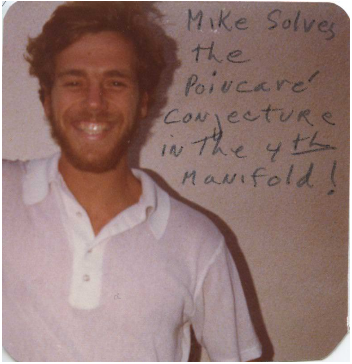
Over the years, dozens of famous mathematicians had announced or published solutions, only to have flaws later discovered.

Steve Smale's higher-dimensional proof

In 1961, Steve Smale surprised the mathematical world by proving every simply connected, closed n-manifold is homeomorphic to the n-sphere, as long as $n > 4$. Poincaré's case of $n = 3$ and the $n = 4$ case remained unproven.

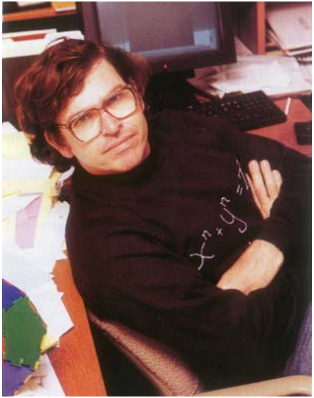
The $n = 4$ case

In 1982 Michael Freedman proved the $n = 4$ dimension.



Bill Thurston's Geometrization Conjecture

Thurston came up with a far-reaching generalization of Poincaré's conjecture. It very roughly said that every closed 3-manifold can be broken up into 8 different types of geometric pieces. This led to new directions in Poincaré research.



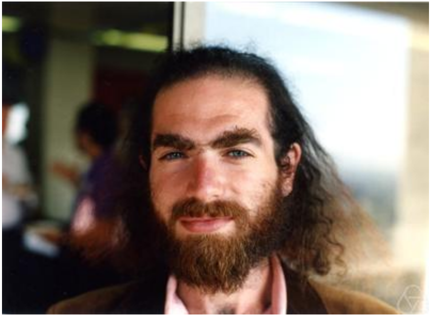
Richard Hamilton's Program

In the early 1980s, Richard Hamilton came up with an approach that would solve the Geometrization Conjecture (and thus Poincaré) if it could be completed. It involved something called the *Ricci flow*. This is an approach using partial differential equations that are related to the famous heat equation. Hamilton made a lot of progress with this approach, but he got hopelessly stuck on some details.

The Clay Mathematics Institute

In 1998, a philanthropist named Landon Clay endowed the Clay Mathematics Institute with a lot of money. In particular, they identified 7 problems as “Millennium Problems” and gave each a $1 million prize. Poincaré’s Conjecture was one of them.

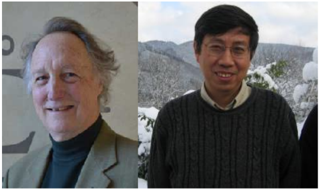
In 2002, Grigory Perelman posted on archive.org (a site where mathematicians and physicists upload drafts of their papers for the public) a paper outlining a solution to the Geometrization Conjecture. He followed this with two more papers. He left out some details, and his arguments were hard to follow, but it was clear that he had done some amazing work.



Verification

Three groups of mathematicians set to working on verifying the details in Perelman's work.

John Morgan & Gang Tian, Huai-Dong Cao & Xi-Ping Zhu, Bruce Kleiner & John Lott



ASIAN J. MATH.
Vol. 10, No. 2, pp. 165–492, June 2006

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A COMPLETE PROOF OF THE POINCARÉ AND GEOMETRIZATION CONJECTURES – APPLICATION OF THE HAMILTON-PERELMAN THEORY OF THE RICCI FLOW*

HUAI-DONG CAO† AND XI-PING ZHU‡

Abstract. In this paper, we give a complete proof of the Poincaré and the geometrization conjectures. This work depends on the accumulative works of many geometric analysts in the past thirty years. This proof should be considered as the crowning achievement of the Hamilton-Perelman theory of Ricci flow.

Key words. Ricci flow, Ricci flow with surgery, Hamilton-Perelman theory, Poincaré Conjecture, geometrization of 3-manifolds

AMS subject classifications. 53C21, 53C44

Story time with uncle Brian

The three groups were filling in the gaps in Perelman's arguments. Cao and Zhu in particular rewrote a lot of the details themselves, especially where Perelman was unclear. But their paper didn't give enough credit to Perelman, and their advisor, Shing-Tung Yau, talked it up in the media as if Cao and Zhu were the ones who had solved the conjecture. This didn't go over very well.

In 2006, Perelman was awarded the Fields Medal, the mathematical equivalent of the Nobel prize. But he turned it down, saying

> *"I'm not interested in money or fame; I don't want to be on display like an animal in a zoo. I'm not a hero of mathematics. I'm not even that successful; that is why I don't want to have everybody looking at me."*

The $1 million prize

In 2010, the Clay Mathematics Institute offered him the $1 million prize. Perelman again turned it down. He felt that the prize should be shared with Hamilton, and he also said,

> *"[the prize] was completely irrelevant for me. Everybody understood that if the proof is correct, then no other recognition is needed.... the main reason is my disagreement with the organized mathematical community. I don't like their decisions, I consider them unjust."*

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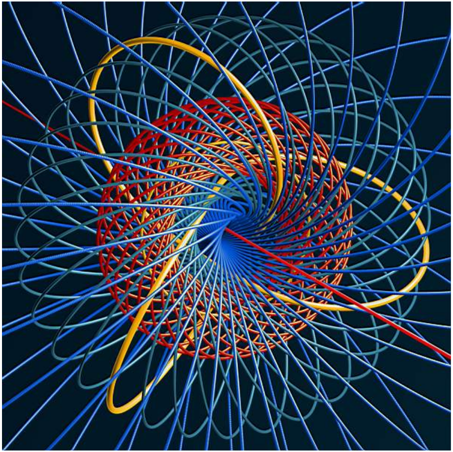
Perelman has stayed out of the mathematical spotlight, living in St. Petersburg and not publicly working on math.

The future

Though the problem has definitively been solved, the various techniques invented for trying to solve it have opened up vast new areas of research.



Further reading

- Article in the New Yorker: *Manifold Destiny* by Sylvia Nasar and David Gruber. `http://www.newyorker.com/magazine/2006/08/28/manifold-destiny`
- Book: *Poincaré's Prize: The Hundred-Year Quest to Solve One of Math's Greatest Puzzles* by George Szpiro, 2007.
- Book: *The Poincaré Conjecture: In Search of the Shape of the Universe* by Donal O'Shea, 2007.

Image credits

- Sphere: https://tex.stackexchange.com/questions/54193/how-to-draw-a-shaded-sphere
- Torus: https://en.wikipedia.org/wiki/Solid_torus
- Earth: http://www.esri.com/news/arcuser/0610/nospin.html
- Wireframe sphere https://en.wikipedia.org/wiki/File:Sphere_wireframe_10deg_6r.svg
- Donut to coffee cup: https://prateekvjoshi.com/2014/11/16/homomorphism-vs-homeomorphism/
- 3-sphere: https://en.wikipedia.org/wiki/File:Hypersphere.png
- Young Poincaré: http://www.annales.org/archives/x/poincare.html
- Old Poincaré: https://commons.wikimedia.org/wiki/File:Henri-Poincar%C3%A9.jpg
- Smale: http://tetrahedral.blogspot.com/2011/03/6-months-in-rio-smale-solution-to.html
- Freedman: http://celebratio.org/viewer/34/
- Thurston: https://blogs.scientificamerican.com/cross-check/files/2012/08/Thurston_2.jpeg
- Hamilton: https://en.wikipedia.org/wiki/File:Richard_Hamilton.jpg
- Morgan: http://www.scgp.stonybrook.edu/wp-content/uploads/2011/04/John-Morgan1.jpg
- Tian: https://en.wikipedia.org/wiki/File:Gang_Tian.jpeg
- Cao and Zhu: http://www.gettyimages.com/photos/conjecture
- Kleiner: https://www.simonsfoundation.org/wp-content/uploads/2015/07/Kleiner-200x300_N4A2693.jpg
- Lott: https://en.wikipedia.org/wiki/John_Lott_(mathematician)
- Perelman: https://en.wikipedia.org/wiki/Grigori_Perelman
- Manifold in last slide: http://galileo.math.siu.edu/mikesullivan/Courses/532/Sum13/