The Sum Choice Number of $P_3 \,\square\, P_n$

Brian Heinold
heinold@msmary.edu
Department of Mathematics and Computer Science
Mount St. Mary's University
Emmitsburg, MD 21727

Abstract

A graph G is said to be f-choosable if there exists a proper coloring from every assignment of lists of colors to the vertices of G where the list sizes are given by f. The sum choice number of G is the minimum $\sum_{v \in V(G)} f(v)$ over all f such that G is f-choosable. Here we determine the sum choice of the cartesian product $P_3 \,\square\, P_n$ to be $8n - 3 - \lfloor n/3 \rfloor$. The techniques used here have applicability to choosability of other graphs.

1 Introduction

Sum list coloring is a type of list coloring where each vertex is assigned a list of colors and one seeks the minimum sum of the list sizes such that, regardless of the lists of those sizes used, there exists a proper coloring from the lists. It is equivalent to minimizing the average list size. Sum list coloring was introduced by Isaak in [6] and [7]. Subsequent work can be found in [1], [2], [3], [4], and [5]. Formally, a *size function* f on a graph G is a function $f : V(G) \to \mathbb{Z}$ assigning each vertex a list size. An *f-assignment* $\mathcal{C}$ is a function assigning each vertex of G a list of colors such that $|\mathcal{C}(v)| = f(v)$ for each $v \in V(G)$. Our colors will be positive integers and a list such as $\{1, 2, 3\}$ will be written in abbreviated form as 123. A *$\mathcal{C}$-coloring* c is a function that assigns each vertex a color such that $c(v) \in \mathcal{C}(v)$ for each $v \in V(G)$. The coloring is *proper* if $c(v) \neq c(w)$ whenever v is adjacent to w. We say G is *f-choosable* if G can be properly colored from every f-assignment. The *sum choice number*, $\chi_{\mathrm{SC}}(G)$, of a graph G is the smallest constant k for which there exists an f such that G is f-choosable and $\sum_{v \in V} f(v) = k$. We denote $\sum_{v \in V} f(v)$ by size(f). If G is f-choosable, f is called a *choice function* for G, and if f is a choice function where size$(f) = \chi_{\mathrm{SC}}(G)$, then f is called a *minimum choice function*.

It is easy to show that the sum choice number of any graph G is bounded by $|V(G)| + |E(G)|$, the *greedy bound* (denoted GB). There are a number of graphs for which equality holds, including complete graphs, paths, and cycles. See [7]. Such graphs are said to be *sc-greedy*. In [4] the author showed that the cartesian product $P_2 \square P_n$ is sc-greedy. We will show below that $\chi_{\mathrm{SC}}(P_3 \square P_n) = \mathrm{GB} - \lfloor n/3 \rfloor$. The techniques we develop to accomplish this should be useful in determining the choosability of other graphs. Section 2 below introduces further notation and provides examples to motivate these techniques. The techniques are formally developed in section 3, and we then apply them to $P_3 \square P_n$ in section 4.

2 Notation and Examples

Let G be a graph, H be an induced subgraph of G, and f be a size function on G. Define the size function f^H on $G - H$ by $f^H(v) = f(v) - |N(v) \cap H|$ for each $v \in V(G - H)$. If H consists of a single vertex $\{w\}$, we will write f^w. We will use f_H to refer to the restriction of f to H, and further, if we say that f is a size function on H, it will be understood that we mean the restriction of f to H.

We label the vertices of $P_3 \square P_n$ as in Figure 1. Let Co_i denote the subgraph induced by the vertices of column i, namely, $v_{1,i}$, $v_{2,i}$, and $v_{3,i}$. Let To_i be the subgraph induced by the top two vertices of column i, namely $v_{1,i}$ and $v_{2,i}$, and let Bo_i be the subgraph induced by the bottom two vertices, $v_{2,i}$ and $v_{3,i}$.

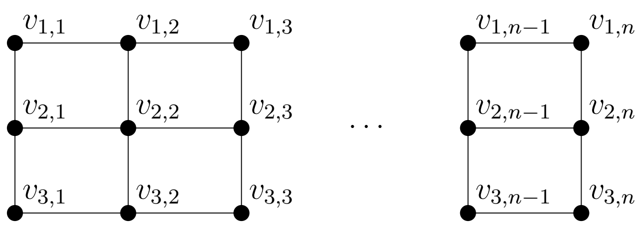

Figure 1: $P_3 \square P_n$

A size function on $P_3 \square P_n$ will be given in array notation, with the (i, j) entry being $f(v_{i,j})$. Figure 2 shows some notational shortcuts we will use. A thin box like the one shown on the left indicates a combined list size of 7 on a column. A box twice as thick, like the one second from the left, indicates a combined list size of 13 on two adjacent columns. The middle box indicates a combined list size of one less than the sum choice number on a collection of adjacent columns. The other two parts of the figure display further examples of notation and should be self-explanatory.

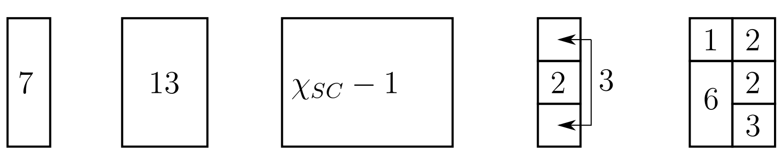

Figure 2: Examples of notation

We can think of list coloring as a game in which someone gives us a size function f and we have to come up with lists to defeat it, that is, lists that show the graph is not f-choosable. Below we have some examples in which we demonstrate our techniques for defeating a given size function.

Example 1: Consider the following size function on $P_3 \square P_2$:

$$f = \begin{matrix} 2 & 2 \\ 1 & 3 \\ 2 & 2 \end{matrix}.$$

The key here to defeating f is that $f(v_{2,1}) = 1$. The most logical approach would be to make sure that the single color in the list for $v_{2,1}$ appears in the list of each neighbor of $v_{2,1}$. Since this color will not be available on any of these neighbors, we have essentially reduced the problem to showing that $P_3 \square P_2 - v_{2,1}$ is not $f^{v_{2,1}}$-choosable (where $f^{v_{2,1}} = {\scriptstyle\begin{matrix} 1 & 2 \\ & 2 \\ 1 & 2 \end{matrix}}$). This, in fact, works in both directions, and it is easy to show for any graph G and size function f where some vertex v satisfies $f(v) = 1$, that G is f-choosable if and only if $G - v$ is f^v-choosable. See Lemma 7 of [7] for a proof of a more general statement.

Example 2: The above idea generalizes. Consider the following size function on $P_3 \square P_3$:

$$f = \begin{matrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 2 \end{matrix}.$$

Look specifically at the restriction of f to the first column, Co_1. There is an f-assignment on Co_1, $\mathcal{C} = {\scriptstyle\begin{matrix} 1 \\ 12 \\ 23 \end{matrix}}$, that has only one proper coloring, c. The entire column plays the same role here that the vertex with list size 1 played in the previous example. Therefore, to defeat the size function we should use $\mathcal{C}$ on Co_1 and choose lists on Co_2 such that $c(v_{i,1})$ is in the list for $v_{i,2}$, $i = 1, 2, 3$. This essentially reduces the problem to showing that $P_3 \square P_3 - Co_1$ is not f^{Co_1}-choosable (where $f^{Co_1} = {\scriptstyle\begin{matrix} 1 & 2 \\ 1 & 3 \\ 2 & 2 \end{matrix}}$).

Example 3: Let us continue generalizing. Consider the following:

$$f = \begin{matrix} 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 2 \\ 2 & 3 & 2 & 2 \end{matrix} \qquad \mathcal{C} = \begin{matrix} 13 & 234 & 46 & 56 \\ 12 & 126 & 456 & 45 \\ 23 & 135 & 56 & 46 \end{matrix}.$$

The lists on $Co_3 \cup Co_4$ are constructed so that any proper coloring from $\mathcal{C}$ on these columns must use color 4 on $v_{1,3}$, color 6 on $v_{2,3}$ and color 5 on $v_{3,3}$. These colors are not available on the neighboring vertices of Co_2 and hence the lists on $Co_1 \cup Co_2$ are essentially reduced to ${\scriptstyle\begin{matrix} 13 & 23 \\ 12 & 12 \\ 23 & 13 \end{matrix}}$, a standard list coloring example which has no proper coloring (see the following example). In this case column 3 plays a similar role to that of column 1 in the previous example. The difference here is that we need help from the lists of column 4 to break column 3 down to having only one proper coloring.

Example 4: In explaining why there are no proper colorings for the lists mentioned in the previous example, we are led to one further generalization of our initial idea. Arrange the lists sideways as below:

$$\begin{array}{ccc} 13 & 12 & 23 \\ 23 & 12 & 13 \end{array}.$$

The list assignment $\substack{12\\12}$ on Co_2 has exactly two proper colorings, c_1 and c_2, where $c_1(v_{1,2}) = 1$, $c_1(v_{2,2}) = 2$, and $c_2(v_{1,2}) = 2$, $c_2(v_{2,2}) = 1$. The lists on Co_1 are constructed so that $c_1(v_{1,2})$ and $c_1(v_{2,2})$ are in the lists for $v_{1,1}$ and $v_{2,1}$, respectively, and the lists for Co_3 are similarly constructed based on c_2. Essentially what is happening is we have a subgraph H where the lists are such that there are two proper colorings of H and there are two disjoint subgraphs which are not f^H-choosable.

3 Techniques and Lemmas

We will now formalize what we have seen in the preceding examples.

Definition 1. *Let G be a graph with an induced subgraph H, and let f be a size function on G. Define $\mathrm{pc}(G, f, H)$ to be the minimum k such that there exists an f-assignment $\mathcal{C}$ for which there are k proper $\mathcal{C}$-colorings of H such that every proper $\mathcal{C}$-coloring of G restricts to one of them. We will use the shorthand $\mathrm{pc}(G, f) = \mathrm{pc}(G, f, G)$*

For instance, in Example 2, $\mathrm{pc}(Co_1, f) \leq 1$ because the lists $\mathcal{C} = \substack{1\\12\\23}$ have exactly one proper coloring (we actually have $\mathrm{pc}(Co_1, f) = 1$ because Co_1 is f-choosable). In Example 3, $\mathrm{pc}(Co_3 \cup Co_4, f, Co_3) \leq 1$. This can be seen from the lists $\substack{46\\456\\56}\,\substack{56\\45\\46}$. We have, however, that $\mathrm{pc}(Co_3, f) > 1$. The idea is that we need to use lists on both Co_3 and Co_4 in order to reduce the number of possible proper colorings on Co_3. Finally, in Example 4, the lists $\substack{12\\12}$ on Co_2 show $\mathrm{pc}(Co_2, f) \leq 2$.

We now give a few lemmas. We will use the notation $c(N(v))$ to denote the set of colors used by the coloring c on the neighbors of vertex v.

Lemma 1. *Let G be a graph with H an induced subgraph of G, let J be an induced subgraph of H, and let f be a size function on G. Suppose there exists an f-assignment $\mathcal{C}$ such that $\mathrm{pc}(H, f) > 0$ and the restriction to J of every proper $\mathcal{C}$-coloring is the same proper coloring, c. Suppose for every v in $V(G - H)$ that $|c(N_J(v))| = |N_J(v)|$. Then $\mathrm{pc}(G, f, G - H) \leq \mathrm{pc}(G - H, f^J)$. Equality holds if $J = H$ and $\mathrm{pc}(H, f) = 1$.*

Proof. Let $\mathcal{D}$ be an f^J-assignment on $G - H$ having $\mathrm{pc}(G - H, f^J)$ proper colorings with the colors named so that no colors used in the lists of $\mathcal{C}$ are used in the lists of $\mathcal{D}$. Extend $\mathcal{C}$ to an f-assignment on all of G by defining $\mathcal{C}(v) = \mathcal{D}(v) \cup c(N_J(v))$ for $v \in V(G - H)$. Under these lists $G - H$ must be colored from $\mathcal{D}$, so $\mathrm{pc}(G, f, G - H) \leq \mathrm{pc}(G - H, f^J)$.

We now show equality when $J = H$ and $\mathrm{pc}(H, f) = 1$. Let $\mathcal{F}$ be any f-assignment. Because $\mathrm{pc}(H, f) > 0$, H can be properly colored by some $\mathcal{F}$-coloring, c'. Consider the list assignment $\mathcal{D}$ on $G - H$ given by $\mathcal{D}(v) = \mathcal{F}(v) - c'(N(v))$. We have $|\mathcal{D}(v)| \geq f^H(v)$ for each $v \in V(G - H)$, so there exist at least $\mathrm{pc}(G - H, f^H)$ proper colorings of $G - H$ from $\mathcal{D}$ and hence at least that many colorings from $\mathcal{F}$. □ □

This lemma formalizes what we saw in Examples 2 and 3. We will use this lemma often enough that it is worthwhile to create notation for it. As shown in Figure 3, we will use an arrow to indicate use of the lemma and gray out the lists used to indicate they cannot be used for further reductions. The left part of the figure corresponds to Example 2, where $H = J = Co_1$. The right part corresponds to Example 3, where $H = Co_3 \cup Co_4$ and $J = Co_3$.

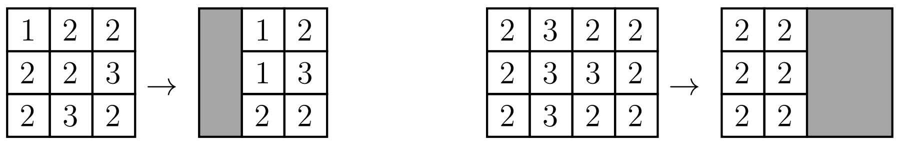

Figure 3: Notation indicating a use of Lemma 1

We will next generalize what we encountered in Example 4:

Lemma 2. *Let G be a graph and f a size function on G. Let H and H_1, H_2, $\ldots$, H_k be induced subgraphs of G whose vertex sets partition $V(G)$, and suppose $H_1, H_2, \ldots, H_k$ are not f^H-choosable. Let $\mathcal{C}$ be an f-assignment on H for which there exist exactly m proper $\mathcal{C}$-colorings $c_1, c_2, \ldots, c_m$. Suppose that for each $i = 1, \ldots, \min\{m, k\}$, there exists an index $j(i)$, with $j(i_1) \neq j(i_2)$ for $i_1 \neq i_2$, such that for each $v \in V(H_{j(i)})$, $|c_i(N_H(v))| = |N_H(v)|$. Then $\mathrm{pc}(G, f, H) \leq \max\{m-k, 0\}$.*

Proof. We will extend $\mathcal{C}$ to all of G so that every proper $\mathcal{C}$-coloring restricts to one of exactly $\max\{m-k, 0\}$ proper colorings on H. By hypothesis, for each $j = 1, \ldots, k$ there exists an f^H-assignment $\mathcal{D}_j$ that has no proper coloring. We can rename the colors if necessary so that these share no colors in common with $\mathcal{C}$. Now extend $\mathcal{C}$ to G as follows: for each $i = 1, 2, \ldots, \min\{m, k\}$ and each $v \in H_{j(i)}$, define $\mathcal{C}(v) = \mathcal{D}_{j(i)} \cup c_i(N_H(v))$. It is clear that if a proper $\mathcal{C}$-coloring f restricts to c_i on H, then $H_{j(i)}$ cannot be properly colored from these lists. Thus there are at most $\max\{m-k, 0\}$ proper $\mathcal{C}$-colorings. □ □

We can picture this with H being in the center and the H_i as appendages that are used to lower the number of proper colorings available for H. When we use this lemma we will specify what the center H is. The verification that the appendages H_i are not f^H-choosable is usually left to the reader. This lemma will be used almost exclusively with $m = 2$ and $k = 1$ or 2.

For the next lemma, recall that paths are sc-greedy; that is, $\chi_{\mathrm{SC}}(P_n) = 2n - 1$. See [7]. This lemma provides some information about size functions near the greedy bound.

Lemma 3. *Let f be a choice function on P_n. If $\mathrm{size}(f) = 2n-1$, then $\mathrm{pc}(P_n, f) = 1$, and if $\mathrm{size}(f) = 2n$, then $\mathrm{pc}(P_n, f) = 2$.*

Proof. The proof of each statement is by induction on n. The base case, $n = 1$, is easy for both statements. Assume now for any g of size $2n-3$ that $\mathrm{pc}(P_{n-1}, g) = 1$ and for any h of size $2n-2$ that $\mathrm{pc}(P_{n-1}, h) = 2$. Let v be an endvertex of the path, and let w be its neighbor on the path.

Suppose first that $f(v) = 1$. If $\mathrm{size}(f) = 2n - 1$, then $\mathrm{size}(f^v) = 2n - 3$, and by the induction hypothesis there exists an f^v-assignment on $P_n - v$ that has only one proper coloring. Since $f(v) = 1$, this f^v-assignment extends to an f-assignment on the entire graph having exactly one proper coloring. A similar argument works when $\mathrm{size}(f) = 2n$.

Suppose next that $\mathrm{size}(f_{P_n - v}) = 2n - 3$. If $\mathrm{size}(f) = 2n - 1$, then by the induction hypothesis there exists an f-assignment on $P_n - v$ which has only one proper coloring, c. Extend this f-assignment to the entire graph by letting the list for v contain $c(w)$. This extension has only one proper coloring. A similar argument works when $\mathrm{size}(f) = 2n$.

The only case left is $\mathrm{size}(f_{P_n - v}) = 2n - 2$ and $f(v) = 2$. In this case, by the induction hypothesis, there exists an f-assignment on $P_n - v$ that has exactly two proper colorings, c_1 and c_2. Extend this to the entire graph by letting the list for v be $\{c_1(w), c_2(w)\}$. This f-assignment has exactly two proper colorings. $\square$ $\square$

This lemma will be most often used in the case of P_3. Specifically, if $\mathrm{size}(f) = 5$, then $\mathrm{pc}(P_3, f) = 1$, and if $\mathrm{size}(f) = 6$, then $\mathrm{pc}(P_3, f) = 2$. We have four more lemmas. The first two are straightforward applications of Lemma 1, so their proofs are omitted. The other two are rather technical lemmas that we will use several times.

Lemma 4. *Let G be a graph with disjoint induced subgraphs G_1 and G_2. Let H_1 and H_2 be induced subgraphs of G_1 and G_2, respectively, with some vertex of H_1 adjacent to a vertex of H_2. Let f be a size function on G such that $\mathrm{pc}(G_1, f, H_1) = 1$ and $\mathrm{pc}(G_2, f, H_2) = 1$. Then G is not f-choosable.*

Lemma 5. *Let f be a size function on $P_3 \square P_n$ such that $\mathrm{pc}(P_3 \square P_n, f, To_{n-1}) = 1$ and $f(v_{1,n}) = f(v_{2,n}) = 2$. Then $P_3 \square P_n$ is not f-choosable. A similar result holds with To_{n-1} and $v_{1,n}$ replaced by Bo_{n-1} and $v_{3,n}$, respectively.*

Lemma 6. *Let f be a choice function on $P_3 \square P_n$. Suppose for any minimum choice function g on $H = P_3 \square P_n - Co_n$ that $\mathrm{pc}(H, g, To_{n-1}) = \mathrm{pc}(H, g, Bo_{n-1}) = 1$.*

(a) Suppose $\mathrm{size}(f_H) = \chi_{\mathrm{SC}}(H)$ and $\mathrm{size}(f_{Co_n}) = 8$. If $f_{Co_n} \neq \begin{smallmatrix} 2 \\ 4 \\ 2 \end{smallmatrix}$, then at least one of $\mathrm{pc}(P_3 \square P_n, f, To_n)$ and $\mathrm{pc}(P_n \square P_3, f, Bo_n)$ equals 1.

(b) Suppose $\mathrm{size}(f_H) = \chi_{\mathrm{SC}}(H) + 1$ and $\mathrm{size}(f_{Co_n}) = 7$. If $f_{Co_n} \neq \begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix}$, then at least one of $\mathrm{pc}(P_3 \square P_n, f, To_n)$ and $\mathrm{pc}(P_n \square P_3, f, Bo_n)$ equals 1.

(c) If $\mathrm{size}(f_H) = \chi_{\mathrm{SC}}(H) + 1$ and $f_{Co_n} = \begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix}$, then $\mathrm{pc}(P_3 \square P_n, f, v_{1,n}) = 1$. More specifically, there exists an f-assignment $\mathcal{C}$ satisfying $\mathcal{C}$ equal to $\begin{smallmatrix} 12 \\ 123 \\ 34 \end{smallmatrix}$ on Co_n from which any proper coloring must use color 2 on $v_{1,n}$. A similar result holds if we replace $v_{1,n}$ by $v_{3,n}$.

Proof. To prove (a), first note that by Lemmas 4 and 5, f cannot assign list size 1 to any vertex of Co_n or size 2 to adjacent vertices of Co_n. Next, if one of $f(v_{1,n})$ and $f(v_{2,n})$ is 2 and the other 3, then the result follows by Lemma 1. An analogous result holds for Bo_n.

To prove (b), consider first $f(v_{i,n}) = 1$ for $i = 1$, 2, or 3. The possibilities (up to symmetry) are taken care of by Lemma 1, as shown in Figure 4 (in the third possibility,

$P_3 \square P_n$ is not f-choosable). On the other hand, if f assigns list size 2 to adjacent vertices of Co_n, then Lemma 2 applies with those two vertices as center.

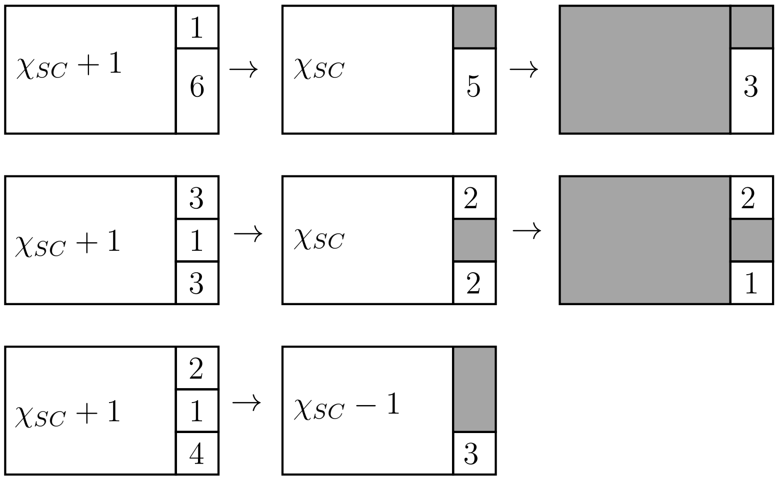

Figure 4: Possibilities for Lemma 6(b)

To prove (c), let g be a size function on $P_3 \square P_n - Co_n$ that agrees with f except that $g(v_{1,n-1}) = f(v_{1,n-1}) - 1$. Then by hypothesis, there is a g-assignment $\mathcal{C}$ such that every proper $\mathcal{C}$-coloring must use color 2 on $v_{2,n-1}$ and color 4 on $v_{3,n-1}$. Name the colors of $\mathcal{C}$ such that color 1 does not appear on $\mathcal{C}$ and extend $\mathcal{C}$ to an f-assignment on $P_3 \square P_n$ by appending color 1 to $\mathcal{C}(v_{1,n-1})$ and defining $\mathcal{C}$ on Co_n to be $\begin{smallmatrix}12\\123\\34\end{smallmatrix}$. Suppose now that a proper coloring uses color 1 on $v_{1,n}$. Then $P_3 \square P_n - Co_n$ must be colored from the original lists that force the proper coloring to use color 2 on $v_{2,n-1}$ and color 4 on $v_{3,n-1}$. But then Co_n could not be properly colored. So any proper coloring from these lists must use color 2 on $v_{1,n}$. $\square$ $\square$

Lemma 7. *Let f be a choice function on $G = P_3 \square P_3 - v_{1,1}$. Suppose $f(v_{2,1}) = f(v_{3,1}) = 2$ and f has size 7 on Co_2 and on Co_3. Let $\mathcal{C}$ be an f-assignment on Bo_1 with $|\mathcal{C}(v_{2,1}) \cap \mathcal{C}(v_{3,1})| = 1$. Then $\mathcal{C}$ can be extended to an f-assignment on all of G such that there is no more than one possible restriction of any proper $\mathcal{C}$-coloring to To_3. The result holds with To_3 replaced by Bo_3.*

Proof. We may assume that $\mathcal{C}(v_{2,1}) = 13$ and $\mathcal{C}(v_{3,1}) = 23$. Suppose first that $f(v_{i,2}) = 1$ for some i. The size of $f^{v_{i,2}}$ on Co_3 is 6 and Lemma 2 with center Co_3 applies. It is not difficult to check that $P_3 \square P_3 - (v_{1,1} \cup v_{i,2} \cup Co_3)$ is not $f^{Co_3 \cup v_{i,2}}$-choosable (and that the lists showing this fit with the requirements of the claim). Next, suppose that $f(v_{i,3}) = 1$ for some i. Then the size of $g = f^{v_{i,3}}$ on Co_2 is 6, so some $f^{v_{i,3}}$-assignment on Co_2 has exactly two proper colorings. Naming the colors so that one of the proper colorings uses color 1 on $v_{2,2}$ and color 2 on $v_{3,2}$ would imply Bo_1 couldn't be properly colored. Thus there is only one possible proper coloring on $Bo_1 \cup Co_2$. We can then use Lemma 1 to conclude the desired result.

The possibilities remaining for f on Co_2 are $\substack{3\\2\\2}$, $\substack{2\\3\\2}$, and $\substack{2\\2\\3}$. The graph is not f-choosable in the first case because the size of f on $Bo_1 \cup Bo_2 \cup Bo_3$ is 12, which is less than the sum choice number ($\chi_{\mathrm{SC}}(P_2 \,\square\, P_3) = 13$ by Theorem 4 in [4]). For the other two possibilities, consider the list assignments $\begin{smallmatrix} & 13\\ 13 & 123\\ 23 & 12\end{smallmatrix}$ and $\begin{smallmatrix} & 23\\ 13 & 12\\ 23 & 123\end{smallmatrix}$ on $Bo_1 \cup Co_2$. In the first case no proper coloring can use color 1 on $v_{2,2}$, so Co_2 must be colored from $\substack{13\\23\\12}$. In the second case no proper coloring can use color 2 on $v_{3,2}$, so Co_2 must be colored from $\substack{23\\12\\13}$. Notice that these list assignments are permutations of each other. So it suffices to assume that the lists for Co_2 are $\substack{23\\12\\13}$ and look at the possible sizes of f on Co_3. Lists for each of the remaining possibilities are shown below and the desired conclusion of the theorem is easily verified in each case. Three dots indicates that a list is irrelevant.

23	13	23	123	23	$\cdots$	23	13	23	13
12	123	12	12	12	23	12	12	12	13
13	23	13	23	13	23	13	123	13	$\cdots$

$\square$ $\square$

4 The sum choice number of $P_3 \,\square\, P_n$

Theorem 8. *The sum choice number of* $P_3 \square P_n$ *is* $\mathrm{GB} - \lfloor n/3 \rfloor$. *Explicitly, it is* $8n - 3 - \lfloor n/3 \rfloor$.

Proof.

Upper Bound

We first prove that $\chi_{\mathrm{SC}}(P_3 \,\square\, P_n) \le \mathrm{GB} - \lfloor n/3 \rfloor$ by exhibiting a choice function of that size. To start, $P_3 \,\square\, P_1$ is a path and is sc-greedy, and $P_3 \,\square\, P_2$ is sc-greedy by Theorem 4 in [4], so there exist choice functions of the appropriate sizes for those cases. Next, we will show that $P_3 \,\square\, P_3$ is f-choosable for $f = \begin{smallmatrix} 2 & 3 & 2\\ 2 & 2 & 3\\ 2 & 2 & 2\end{smallmatrix}$.

This requires a bit of work. First, we show for any f-assignment $\mathcal{C}$ on $H = Bo_1 \cup Bo_2$, that there must exist at least two proper colorings c_1 and c_2 such that either $c_1(v_{2,1}) \ne c_2(v_{2,1})$ or $c_1(v_{3,2}) \ne c_2(v_{3,2})$. Suppose that every proper $\mathcal{C}$-coloring uses the same color on $v_{2,1}$. We may assume $\mathcal{C}(v_{2,1}) = 12$ and the color used is 1. Then the lists must be of the form shown below on the left. There are, up to symmetry, two possible cases from this, and they are shown below. It can be easily checked that there is a proper coloring using each color on $v_{3,2}$.

12	$2a$	12	21	12	23
$2b$	ab	23	13	24	34

Now consider lists $\mathcal{C}$ on the entire graph. By symmetry, using the result of the previous paragraph, we may suppose that there exist two proper $\mathcal{C}$-colorings p and p' of H with $p(v_{2,1}) \ne p'(v_{2,1})$. Set $x = p(v_{2,1})$, $y = p(v_{2,2})$, and $z = p(v_{3,2})$. We can extend this to a proper coloring of the entire graph unless the lists on the path $G - H$ are of the form

$\mathcal{C}(v_{1,1}) = xa$, $\mathcal{C}(v_{1,2}) = yab$, $\mathcal{C}(v_{1,3}) = bc$, $\mathcal{C}(v_{2,3}) = ycd$, $\mathcal{C}(v_{3,3}) = zd$. In addition, setting $x' = p'(v_{2,1})$, $y' = p'(v_{2,2})$ and $z' = p'(v_{3,2})$, we have that p' can be extended to a proper coloring of the entire graph unless the lists on the path $G - H$ are of the form $\mathcal{C}(v_{1,1}) = x'a'$, $\mathcal{C}(v_{1,2}) = y'a'b'$, $\mathcal{C}(v_{1,3}) = b'c'$, $\mathcal{C}(v_{2,3}) = y'c'd'$, $\mathcal{C}(v_{3,3}) = z'd'$.

Now as $x' \neq x$, we can see $x' = a$ and $a' = x$. As $x = a' \in \mathcal{C}(v_{1,2}) = yab$ and $x \neq a$, $x \neq y$, we must have $x = b$. So $a' = b$ and since $b' \neq a'$, we have $b' \neq b$. Next, since $b' \neq b$ and $b' \in \mathcal{C}(v_{1,3}) = bc$, we have $b' = c$ and $c' = b$. Also, because $b' \neq b$, $b' = c \neq y$, and $b' \in \mathcal{C}(v_{1,2}) = yab$, we have $b' = a$ and so $c = a$. Then, since $yab{=}y'a'b'$, and $b' = a$, $a' = b$, we have $y' = y$. As $y'c'd' = ycd$, and $c = a$, $c' = b$, the list on $v_{2,3}$ must equal yab, and as $d \neq c = a$, we must have $d = b$. Further, as $d' \neq c' = b$, we have $d' = a$. Finally, as $b = d \in \mathcal{C}v_{3,3}$ and $a = d' \in \mathcal{C}(v_{3,3})$, the list on $v_{3,3}$ must equal ab. All of this means the lists must be as shown below (a dot indicates an unknown color).

$$\begin{array}{ccc} ab & yab & ab \\ ab & y\,\cdot & yab \\ \cdot\,\cdot & ab & ab \end{array}$$

We can find a proper coloring from these lists by using color b on $v_{2,1}$ and color y on $v_{2,2}$. Tracing the implications from these choices allows us to properly color all of the vertices, except possibly $v_{3,1}$. However, as this coloring uses color b on both neighbors of $v_{3,1}$, there is a color available there.

Having established this, we now inductively obtain a minimum choice function in the general case. For $n \equiv 0 \pmod 3$ we obtain a choice function of size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 7$ as follows: Set $G = P_3 \,\square\, P_n$ and $H = Co_{n-2} \cup Co_{n-1} \cup Co_n$. Given a minimum choice function g on $P_3 \,\square\, P_{n-3}$ define a size function f to be equal to $\begin{smallmatrix} 3&3&2 \\ 4&2&2 \\ 3&2&2 \end{smallmatrix}$ on H and set $f = g$ on $G - H$. To see that G is f-choosable, let $\mathcal{C}$ be a f-assignment. There exists a proper $\mathcal{C}$-coloring of $G - H$, and notice that f^{G-H} is equal to the choice function of size 20 considered above, so we will be able to color H as well.

For $n \equiv 1, 2 \pmod 3$, we can extend a minimum choice function on $P_3 \,\square\, P_{n-1}$ to $P_3 \,\square\, P_n$ by assigning sizes $\begin{smallmatrix} 3 \\ 2 \\ 3 \end{smallmatrix}$ on Co_n. Calling the extended function f, notice that $f^{G-Co_n} = \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$, so there exists a proper coloring of Co_n from any lists of these sizes. So we can combine such a proper coloring with a proper coloring $G - Co_n$ to color all of G. We thus obtain a choice function of size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 8$ on $P_3 \,\square\, P_n$.

Lower bound

We will show by strong induction on n that

$$\chi_{\mathrm{SC}}(P_3 \,\square\, P_n) \geq \chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + \begin{cases} 7 & \text{if } n \equiv 0 \pmod 3 \\ 8 & \text{otherwise} \end{cases},$$

and moreover that any minimum choice function f on $P_3 \,\square\, P_n$ satisfies that if $n \equiv 0$ or $1 \pmod 3$, then $\mathrm{pc}(P_3 \,\square\, P_n, f, Co_n) = 1$, and if $n \equiv 2 \pmod 3$, then $\mathrm{pc}(P_3 \,\square\, P_n, f, To_n) = 1$ and $\mathrm{pc}(P_3 \,\square\, P_n, f, Bo_n) = 1$. We will call this the *minimum choice property*. From this the statement of the theorem follows.

For all that follows assume that f is a size function on $P_3 \,\square\, P_n$. Our base cases are $n = 1, 2$. The $n = 1$ case follows directly from Lemma 3. Next is the $n = 2$ case. Following that will be three cases according to the congruence of n modulo 3. Each case will consist of first verifying the lower bound, which is done by showing there is no choice function of size one less than the proposed value, and then verifying the minimum choice property.

Case $n = 2$**:**

Let f be a minimum choice function on $P_3 \,\square\, P_2$. We must show $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{To}_2) = \mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{Bo}_2) = 1$. Note that by symmetry this result will also hold for To_1 and Bo_1. Now by Theorem 4 in [4], $\chi_{\mathrm{SC}}(P_2 \square P_3) = 13$ and since $\chi_{\mathrm{SC}}(P_3) = 5$, there are four possibilities for size(f_{Co_2}): 5, 6, 7, and 8. For size 5, $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{Co}_2) = 1$ by Lemma 3. For size 6, $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{Co}_2) = 1$ by Lemma 2 with center Co_2 (the value of m in Lemma 2 is 2 by Lemma 3). For size 8, $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{Co}_2) = 1$ by Lemma 1. For this part of the paper it is helps to have something to look at, so these cases are shown in the top line of Figure 5.

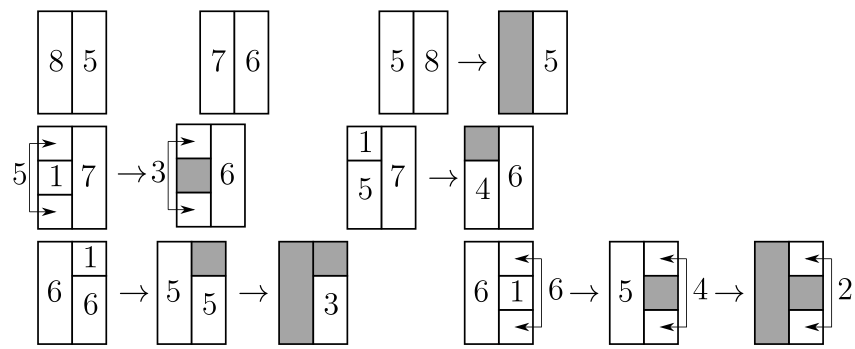

Figure 5: $n = 2$ case

For the size 7 case, see the bottom two lines of Figure 5 for all the possibilities where a list of size 1 occurs. For those for which the size 1 list is on Co_1, Lemma 1 is used followed by Lemma 2 with center Co_2. For those with the size 1 list on Co_2, only Lemma 1 is used. This just leaves the two cases below (up to symmetry). For the first size function, immediately to the right are lists that show $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{Co}_2) = 1$. Shown immediately to the right of the second size function is a set of lists that shows $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{To}_2) = 1$. To show $\mathrm{pc}(P_3 \,\square\, P_2, f, \mathit{Bo}_2) = 1$ use Lemma 2 on these lists with center Bo_2.

2	2	13	23	2	3	23	123
2	3	12	123	2	2	12	12
2	2	23	13	2	2	13	23

Case $n \equiv 0 \pmod 3$**:**

Lower bound: We show that there is no choice function f of size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 6$. The only possibilities for f are shown in the top half of Figure 6 and each is taken care of by Lemma 1.

Minimum choice property: We must show that if f is a minimum choice function on $P_3 \,\square\, P_n$, then $\mathrm{pc}(P_3 \,\square\, P_n, f, \mathit{Co}_n) = 1$. Note that f has size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 7$. The cases to

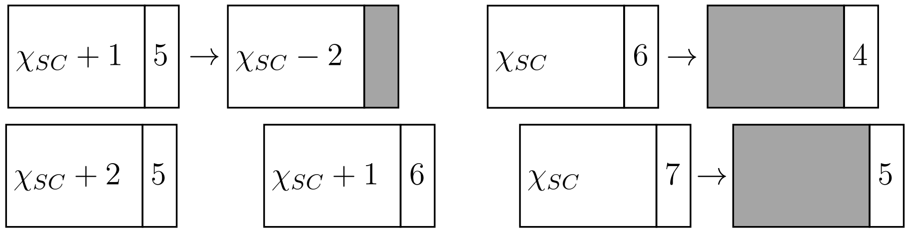

Figure 6: Possibilities for the $n \equiv 0 \pmod 3$ case

consider are shown in the bottom half of Figure 6. The first is clear and the second is taken care of by Lemma 2 with center Co_n. The last case follows quickly from Lemma 1 using the induction hypothesis.

Case $n \equiv 1 \pmod 3$**:**

Lower bound: We show that there is no choice function f of size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 7$. We look at possible sizes on $Co_{n-1} \cup Co_n$. The only possibilities for f are shown in Figure 7 and each is taken care of by Lemma 1.

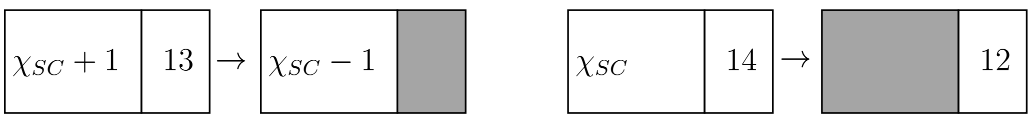

Figure 7: Possibilities for the lower bound of the $n \equiv 1 \pmod 3$ case

Minimum choice property: Let f be a choice function on $P_3 \square P_n$ of size $\chi_{\mathrm{SC}}(P_3 \square P_{n-1})+8$. We must show $\mathrm{pc}(P_3 \,\square\, P_n, f, Co_n) = 1$. We look at the possible sizes on Co_n. The size 5 case is trivial. Size 6 is taken care of by Lemma 2, and size 8 is taken care of by Lemma 1. See the top line of Figure 8.

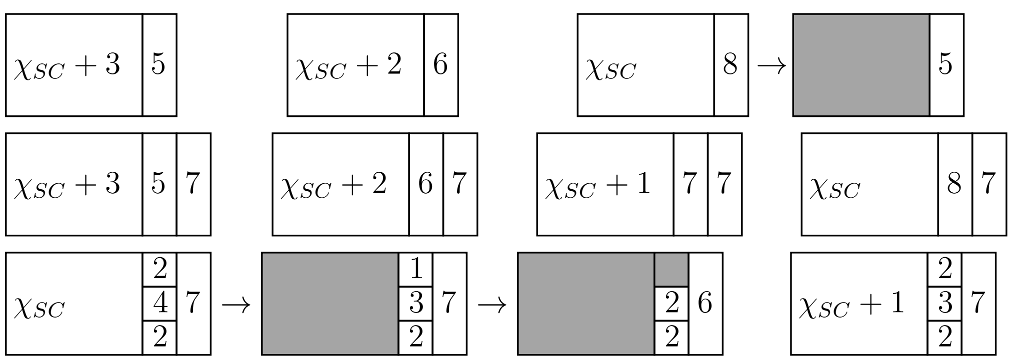

Figure 8: Possibilities for the minimum choice property of the $n \equiv 1 \pmod 3$ case

The remaining case is size 7 on Co_n. See the middle line of Figure 8 for the possibilities on Co_{n-1}. If f has size 5 or 6 on Co_{n-1}, then the graph is not f-choosable by the $n = 2$

case when f has size 5 on Co_{n-1} and by Lemma 2 with center Co_{n-1} when f has size 6 on Co_{n-1}. If f has size 8 on Co_{n-1}, then Lemma 6 (a) takes care of all the possibilities except if f is $\begin{smallmatrix}2\\4\\2\end{smallmatrix}$ on Co_{n-1}. Repeated applications of Lemma 1 followed by an application of Lemma 2 with center Co_n take care of this case, as shown in the bottom left of Figure 8. If f has size 7 on Co_{n-1}, then Lemma 6 (b) together with Lemma 1 take care of everything except the possibility shown in the bottom right of Figure 8. For this possibility, Lemma 6 (c) guarantees that any proper coloring from an f-assignment $\mathcal{C}$ equal to $\begin{smallmatrix}12\\123\\34\end{smallmatrix}$ on Co_{n-1} must use color 2 on $v_{1,n-1}$. Note further that by symmetry we can apply the lemma in essentially the same way but with $v_{3,n-1}$ taking the place of $v_{1,n-1}$. We want extend $\mathcal{C}$ to include Co_n in cases depending what f is on Co_n. Note first that by Lemma 4 if $f(v_{i,n}) = 1$ or if $f(v_{i,n}) = 2$ and $f(v_{2,n}) = 1$ for $i = 1, 3$, then $P_3 \,\Box\, P_n$ is not f-choosable. The lists given below take care of the remaining cases (by symmetry these are the only remaining cases). Each of these is easily checked by starting with the fact that color 2 must be used on $v_{1,n-1}$.

12	123	12	23	12	124
123	1	123	134	123	14
34	124	34	14	34	14

Case $n \equiv 2 \pmod 3$**:**

Lower bound: We show that there is no choice function f of size $\chi_{\mathrm{SC}}(P_3 \,\Box\, P_{n-1}) + 7$. First, the possible sizes on Co_n are 5, 6, and 7. See the top two lines of Figure 9. The first and last cases are easily taken care of by Lemma 1. Next, the possible sizes on $Co_{n-1} \cup Co_n$ are 13, 14, and 15. Size 15 is quickly taken care of by Lemma 1.

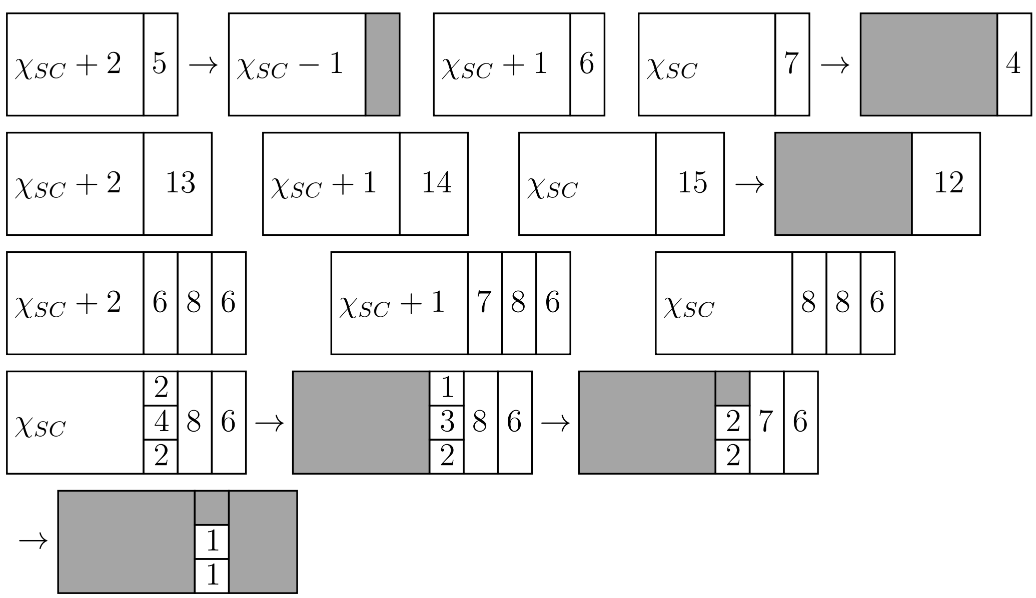

Figure 9: Some possibilities for the lower bound of the $n \equiv 2 \pmod 3$ case

Next, looking at size 14, we must have size 6 on Co_n and size 8 on Co_{n-1}. The possible sizes on Co_{n-2} are 6, 7, and 8. See the third line of Figure 9. The size 6 case follows from Lemma 2 with center Co_{n-2}. For the size 7 case, Lemma 6(b) takes care of all the cases except if f is $\begin{smallmatrix}2\\3\\2\end{smallmatrix}$ on Co_{n-2}. In this case, Lemma 6 (c) guarantees that any proper coloring from an f-assignment $\mathcal{C}$ equal to $\begin{smallmatrix}12\\123\\34\end{smallmatrix}$ on Co_{n-2} must use color 2 on $v_{1,n-2}$. By the $n = 2$ case, there exists an $f^{v_{1,n-2}}$-assignment $\mathcal{D}$ on $Co_{n-1} \cup Co_n$ such that any proper $\mathcal{D}$-coloring must use color 1 on $v_{2,n-1}$ and color 4 on $v_{3,n-1}$. If we extend $\mathcal{C}$ to $Co_{n-1} \cup Co_n$ by setting $\mathcal{C}(v) = \mathcal{D}(v)$ for each vertex v except that we append color 1 to the list of $v_{1,n-1}$, then the resulting lists have no proper coloring. Next, for the size 8 case, Lemma 6 (a) and Lemma 1 take care of all the cases except the one shown on the bottom two lines of Figure 9, for which repeated applications of Lemma 1 apply.

We now look at size 13. See the top line of Figure 10. The possible sizes on Co_{n-2} are 7, 8, and 9. The size 7 and 9 cases are shown in the second and third lines, respectively, of Figure 10. For the size 8 case, Lemma 4 takes care of the case where a list size of 1 appears on Co_{n-2}. The remaining cases up to symmetry are shown on the bottom three lines of Figure 10.

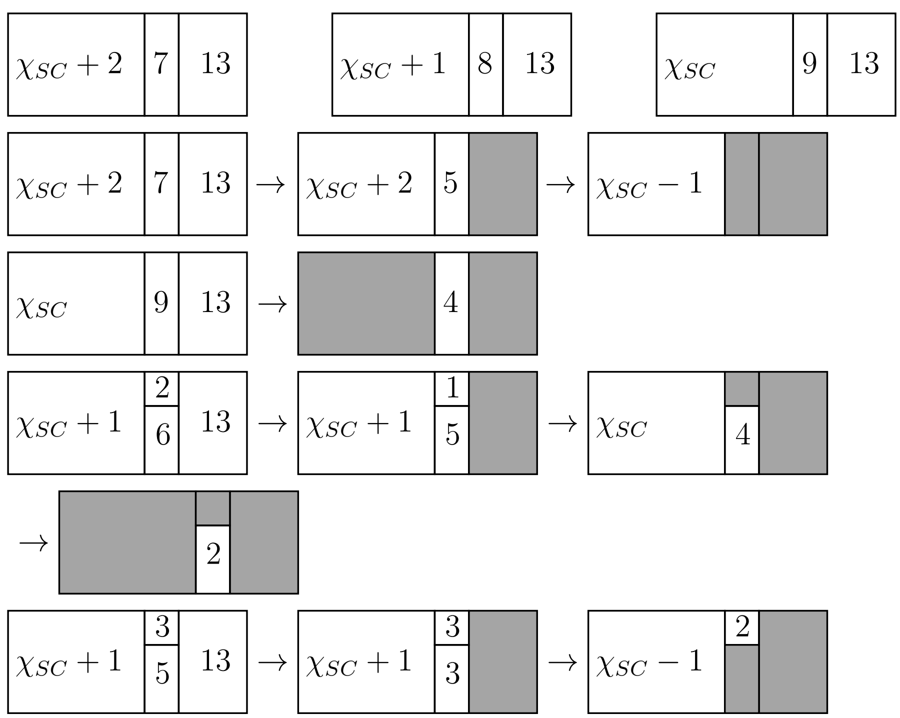

Figure 10: Further possibilities for the lower bound of the $n \equiv 2 \pmod 3$ case

Minimum choice property: Let f be a function of size $\chi_{\mathrm{SC}}(P_3 \,\square\, P_{n-1}) + 8$. The possible sizes on Co_n are 5, 6, 7, and 8. Sizes 5 and 8 are taken care of by Lemma 1 and size 6 follows from Lemma 2 with center Co_n. The possible sizes on $Co_{n-1} \cup Co_n$ are 13, 14, 15, and 16. Sizes 13 is clear and size 16 follows from Lemma 1. See Figure 11.

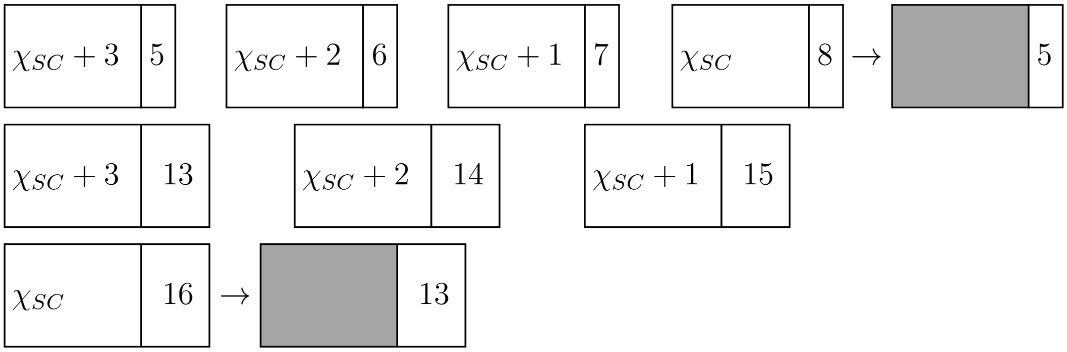

Figure 11: Possibilities for the minimum choice part of the $n \equiv 2 \pmod 3$ case

Now consider the case where the size on $Co_{n-1} \cup Co_n$ is 15. As previously mentioned, the only size of f on Co_n left to consider is size 7. For Co_{n-2}, the possible sizes are 6, 7, and 8. See Figure 12. In the case of size 6, Lemma 2 with center Co_{n-2} implies $P_3 \,\square\, P_n$ is not f-choosable. For size 7, we can apply Lemma 6(b) and Lemma 1 except in the case where f on Co_{n-2} is $\substack{2\\3\\2}$. In that case Lemma 6 (c) guarantees that any proper coloring from an f-assignment $\mathcal{C}$ equal to $\substack{12\\123\\34}$ on Co_{n-2} must use color 2 on $v_{1,n-2}$. Thus color 2 is not available on $v_{1,n-1}$ or $v_{2,n-2}$, so we have satisfied the conditions for Lemma 7 (using $f^{v_{1,n-2}}$), which implies the desired result. Finally, in the case of size 8, we can apply Lemma 6(b) and Lemma 1 except in the case where f on Co_{n-2} is $\substack{2\\4\\2}$. This case is taken care of by Lemma 1 followed by Lemma 7.

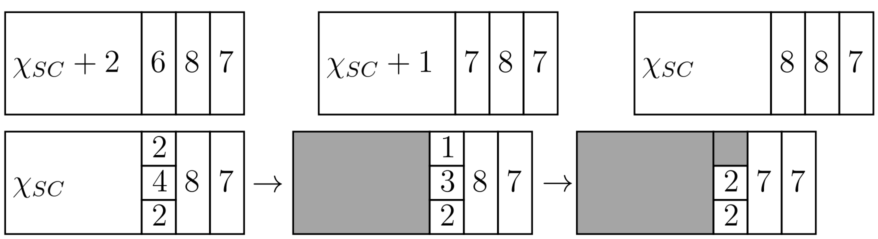

Figure 12: Some possibilities where the size on $Co_{n-1} \cup Co_n$ is 15

Now consider size 14 on $Co_{n-1} \cup Co_n$. Since the size on Co_n must be 7, the size on Co_{n-1} must be 7 also. The possibilities on Co_{n-2} are sizes 5, 6, 7, 8, and 9. Size 5 would lead to a size of 19 on $Co_{n-2} \cup Co_{n-1} \cup Co_n$, which is less than the sum choice number. The other possibilities are shown on the top line of Figure 13. For size 6, we then have that the size on $Co_{n-2} \cup Co_{n-1} \cup Co_n$ must be 20 and the result follows from the $n = 3$ case. For the other cases, first note that if one of the lists on a vertex v of Co_{n-2} has list size 1, then f^v has size 13 on $Co_{n-1} \cup Co_n$ and the result follows from the $n = 2$ case. Next, if f assigns list size 2 to adjacent vertices of Co_{n-2}, then in the case of size 7, Lemma 7 applies, and otherwise $P_3 \,\square\, P_n$ is not f-choosable (in the case of size 8, this follows from Lemma 2 with center those two vertices, and in the case of size 9 it follows from Lemma 5.) The remaining possibilities for size 9 are shown on the bottom three lines of Figure 13. In the first case, the graph turns out not to be f-choosable. The last case requires an application of Lemma 7 after a Lemma 1 reduction.

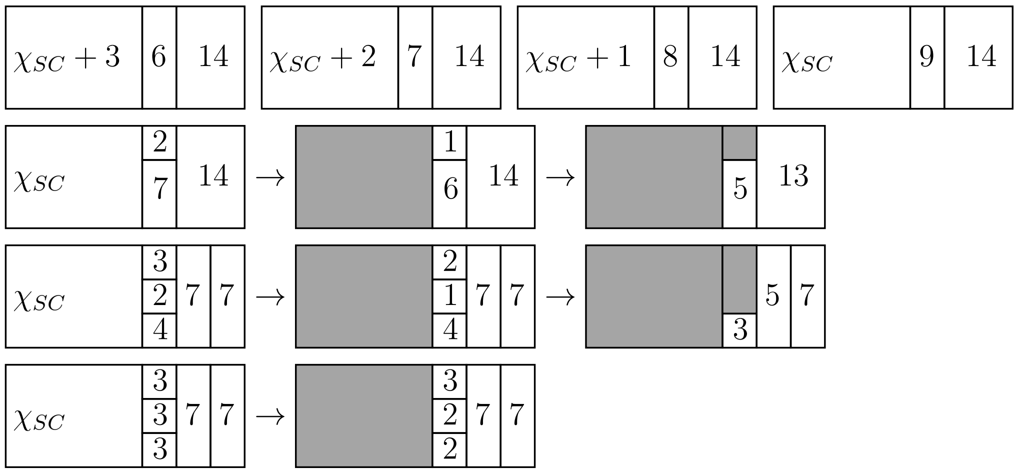

Figure 13: Possibilities where the size on $Co_{n-1} \cup Co_n$ is 14

The only remaining possibility for size 7 is if f is equal to $\begin{smallmatrix}2\\3\\2\end{smallmatrix}$ on Co_{n-2}. Let g be a size function on $P_3 \,\square\, P_n - (Co_{n-2} \cup Co_{n-1} \cup Co_n)$ equal to f everywhere except $g(v_{i,n-3}) = f(v_{i,n-3}) - 1$ for $i = 1, 2, 3$. As g has size less than the sum choice number, let $\mathcal{C}$ be a g-assignment with no proper coloring and choose the color names so that color i is not in $\mathcal{C}(v_{i,n-3})$ for $i = 1, 2, 3$. Extend $\mathcal{C}$ to Co_{n-2} by setting it equal to $\begin{smallmatrix}12\\123\\23\end{smallmatrix}$ there. Then no proper $\mathcal{C}$-coloring can use color 2 on $v_{2,n-2}$. From here Lemma 7 applies. A very similar argument applies in the case of size 8 if f assigns list size 2 to a vertex of Co_{n-2} and list size 3 to an adjacent vertex of Co_{n-2}.

The only remaining case is if f on Co_{n-2} is $\begin{smallmatrix}2\\4\\2\end{smallmatrix}$. Consider cases on Co_{n-1}. If f assigns list size 2 to both vertices of $H = \{v_{1,n-1}, v_{2,n-1}\}$, then Lemma 2 applies with center H. See the top line of Figure 14 for the work needed to show $P_3 \,\square\, P_n - (Co_{n-1} \cup v_{3,n-1})$ is not f^H-choosable. The same argument applies if $H = \{v_{2,n-1}, v_{3,n-1}\}$. Next, if $f(v_{1,n-1}) = 1$ or $f(v_{3,n-1}) = 1$, then $P_3 \square P_n$ is not f-choosable. See the second and third lines of of Figure 14. If $f(v_{2,n-1}) = 1$, then we first apply Lemma 1. See the last line of Figure 14. Then by Lemma 6(c) we can choose lists on $P_3 \square P_n - (Co_{n-1} \cup Co_n)$ so that every proper coloring has only one choice of a color on $v_{1,n-2}$. Choose lists so to make that color unavailable on $v_{1,n-1}$. So, by this point, either $v_{1,n-1}$ or $v_{3,n-1}$ has been reduced to a list size of 1. Therefore we can apply Lemma 1 one more time to reduce down to a list size of 5 on Co_n and the result follows.

Next, consider cases on Co_n. If any vertex gets list size 1, the result follows. See Figure 15. Up to symmetry, the only possibilities left to check are f equal to $\begin{smallmatrix}2\\3\\2\end{smallmatrix}$ on Co_{n-1}, and f equal to $\begin{smallmatrix}2\\3\\2\end{smallmatrix}$ or $\begin{smallmatrix}2\\2\\3\end{smallmatrix}$ on Co_n. Define a size function g on $P_3 \,\square\, P_n - (Co_{n-2} \cup Co_{n-1} \cup Co_n)$ equal to

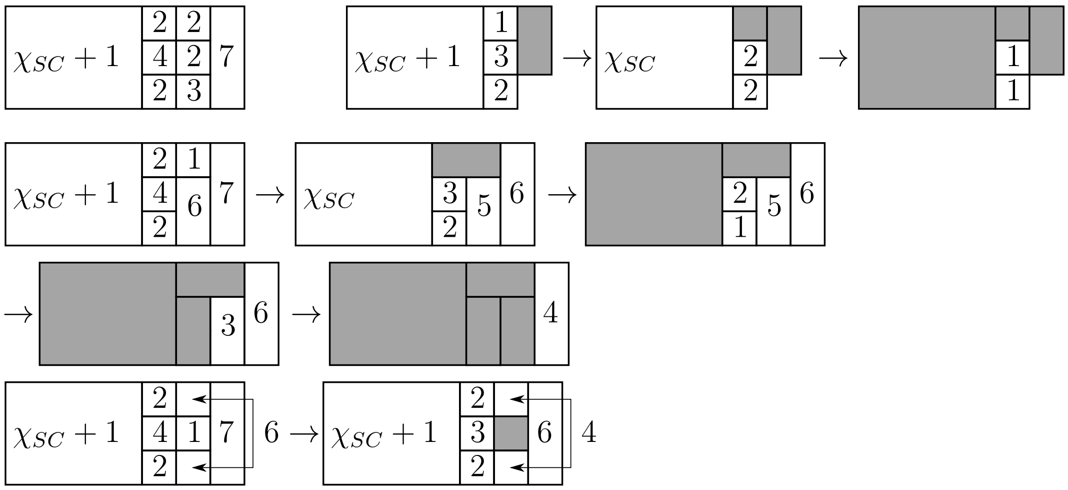

Figure 14: Subcases of the final case

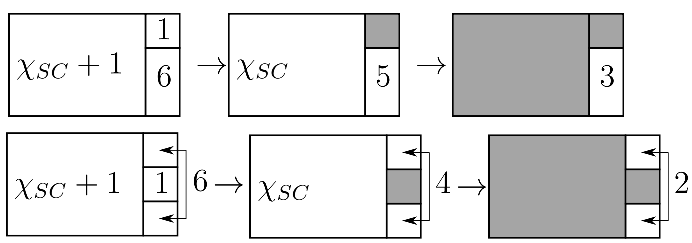

Figure 15: A few more subcases of the final case

f everywhere except $g(v) = f(v) - 1$ for $v \in To_{n-3}$. Since the size of g is less than the sum choice number, there exists a g-assignment $\mathcal{C}$ that has no proper coloring. Assume that the colors are named so that color 1 is not in $\mathcal{C}(v_{1,n-3})$ and color 2 is not in $\mathcal{C}(v_{2,n-3})$ and append those colors to the respective vertices to turn $\mathcal{C}$ into an f-assignment. Next, define the following list assignments, where x can be any color other than 2 or 3.

$$\mathcal{D}_1 = \begin{array}{ccc} 12 & 23 & 13 \\ 1234 & 123 & 12 \\ 14 & 13 & 23x \end{array} \qquad \mathcal{D}_2 = \begin{array}{ccc} 12 & 23 & 13 \\ 1234 & 123 & 123 \\ 14 & 13 & 23 \end{array} .$$

In the case of f equal to $\begin{smallmatrix}2\\2\\3\end{smallmatrix}$ on Co_n extend $\mathcal{C}$ to all of $P_3 \square P_n$ by setting $\mathcal{C}$ equal to $\mathcal{D}_1$ on $Co_{n-2} \cup Co_{n-1} \cup Co_n$. No proper $\mathcal{C}$-coloring can use color 2 on $v_{2,n-2}$ as otherwise $P_3 \square P_n - (Co_{n-2} \cup Co_{n-1} \cup Co_n)$ would have to be colored by non-colorable lists. Our goal is to show $\mathrm{pc}(P_3 \square P_n, f, To_n) = 1$ and $\mathrm{pc}(P_3 \square P_n, f, Bo_n) = 1$. The former does not rely on any of this as it follows from Lemma 2 with center To_n. For the latter we must have that any proper $\mathcal{C}$-coloring c must satisfy $c(v_{2,n}) = 2$ and color $c(v_{3,n}) = x$. This can be easily checked by first supposing that $c(v_{2,n}) = 1$, tracing through to get a contradiction, then supposing that $c(v_{3,n}) = 3$, and tracing through using the fact that $c(v_{2,n}) = 2$ to get a contradiction.

Finally, consider f equal to $\begin{smallmatrix}2\\3\\2\end{smallmatrix}$ on Co_n. Extend $\mathcal{C}$ to all of $P_3 \square P_n$ by setting $\mathcal{C}$ equal to $\mathcal{D}_2$ on $Co_{n-2} \cup Co_{n-1} \cup Co_n$. Again, no proper $\mathcal{C}$-coloring can use color 2 on $v_{2,n-2}$. We show that $\mathrm{pc}(P_3 \square P_n, f, Co_n) = 1$. It suffices to verify that any proper $\mathcal{C}$-coloring c must satisfy $c(v_{1,n}) = 1$, $c(v_{2,n}) = 3$, and $c(v_{3,n}) = 2$. One can check this by first supposing that $c(v_{2,n}) = 1$, tracing through to get a contraction, and then supposing that $c(v_{2,n}) = 2$, tracing through again to get a contradiction. $\square$ $\square$

5 Conclusion

Though the analysis of $P_3 \square P_n$ has proved to be tedious, it has hopefully demonstrated how the techniques developed earlier in the paper are used. The author has attempted to apply these techniques to larger cartesian products, $P_m \square P_n$, but has not been able to fully determine the sum choice number even in the case $m = 4$. It would be interesting to see if the techniques above are sufficient or if something new is needed.

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References

[1] A. Berliner, U. Bostelmann, R.A. Brualdi, L. Deaett, Sum list coloring graphs, *Graphs Combin.*, **22** (2006), no. 2, 173-183.

[2] B. Heinold, PhD Thesis, Lehigh University, May 2006.

[3] B. Heinold, A survey of sum list coloring, *Graph Theory Notes of New York*, **LII** (2007), 38-44.

[4] B. Heinold, Sum choice numbers of some graphs. *Discrete Mathematics*, **309** (2009), 2166-2173.

[5] Z. Furedi, I. Kantor, List colorings with distinct list sizes, the case of complete bipartite graphs. *Electronic Notes in Discrete Mathematics* **34**, (2009), 323-327.

[6] G. Isaak, Sum list coloring $2 \times n$ arrays, *Elec. J. Combin.*, **9** (2002), Note 8.

[7] G. Isaak, Sum list coloring block graphs, *Graphs Combin.*, **20** (2004), 499-506.