Sum Choice Numbers of Some Graphs

Brian Heinold
heinold@msmary.edu
Department of Mathematics and Computer Science
Mount St. Mary's University
Emmitsburg, MD 21727

Abstract

Let f be a function assigning list sizes to the vertices of a graph G. The sum choice number of G is the minimum $\sum_{v\in V(G)} f(v)$ such that for every assignment of lists to the vertices of G, with list sizes given by f, there exists proper coloring of G from the lists. We answer a few questions raised in a paper of Berliner, Bostelmann, Brualdi, and Deaett. Namely, we determine the sum choice number of the Petersen graph, the cartesian product of paths $P_2 \,\square\, P_n$, and the complete bipartite graph $K_{3,n}$.

1 Introduction

List coloring is a form of graph coloring in which each vertex is given a list of permissible colors, and one tries to assign colors to vertices such that each vertex is assigned a color from its list, with adjacent vertices getting different colors. More formally, our setting is as follows: We have a graph G with vertex set V and a set of colors C. Usually we take C to be a finite set of positive integers, and lists such as $\{1,2,3\}$ are written in the abbreviated form 123. A *size function* $f : V \to \mathbb{Z}$ assigns to each vertex a list size. An *f-assignment* $\mathcal{C} : V \to 2^{\mathrm{C}}$ is an assignment of a list of colors to each vertex v such that $|\mathcal{C}(v)| = f(v)$. A *$\mathcal{C}$-coloring* is a function $c : V \to \mathrm{C}$ such that $c(v) \in \mathcal{C}(v)$, and c is called *proper* if $c(v) \neq c(w)$ when v and w are adjacent vertices. If G has a proper $\mathcal{C}$-coloring we say G is *$\mathcal{C}$-colorable*, or simply that $\mathcal{C}$ is *colorable.* We say G is *f-choosable* if G can be properly colored from every f-assignment. If G is f-choosable where $f \equiv k$ for some integer k, then G is said to be *k-choosable.* The smallest constant k for which is G is k-choosable, often called the *choice number*, has been a topic of considerable interest.

In this paper we try to minimize the sum of the list sizes. That is, we seek the smallest constant k for which G is f-choosable with $\sum_{v\in V} f(v) = k$. This constant is called the *sum choice number* of the graph, and it is denoted by $\chi_{\mathrm{SC}}(G)$. We further denote $\sum_{v\in V} f(v)$ by $\mathrm{size}(f)$. Showing $\chi_{\mathrm{SC}}(G) = k$ proceeds in two parts. We must exhibit a function f of size k such that every

f-assignment has a proper coloring, and for every g of size $k-1$, we must find a g-assignment with no proper coloring.

We can get an upper bound for the sum choice number as follows. Choose any ordering $v_1, \ldots, v_n$ of the vertices of G. Define a size function f by $f(v_i) = 1 + |\{v_j : i < j, \text{ and } v_iv_j \in E(G)\}|$ for $i = 1, \ldots, n$. Then $\text{size}(f) = n + e$, where n is the number of vertices of G and e is the number of edges. *Greedy coloring*, that is, coloring the vertices in order according to their index such that each vertex is assigned the smallest color on its list which is has not been assigned to a vertex of lower index, shows that G is f-choosable, and hence for any graph, $\chi_{\text{SC}}(G) \leq n + e$. We will refer to $n + e$ as the *greedy bound*, and sometimes denote it by $\text{GB}(G)$, or just GB when there is only one graph involved. Any graph for which the sum choice number is in fact $n+e$ is called *sc-greedy*. Such graphs include trees, cycles, complete graphs. Moreover, if each of the blocks of a graph are sc-greedy, then the graph itself is sc-greedy. See [1] and [5].

A size function f for which G is f-choosable will be called a *choice function*. Let G be a graph with an induced subgraph H. We denote by f_H, $\mathcal{C}_H$, and c_H, the restrictions of the size function, etc. to H. For any vertex $v \in V(G)$, we define the size function $\widetilde{f}_v$ on $G - v$ by $\widetilde{f}_v(w) = f(w) - 1$, if w is adjacent to v, and $\widetilde{f}_v(w) = f(w)$ otherwise. Note that, in general, a size function g may have $g(v) \leq 0$. In this case, any g-assignment $\mathcal{C}$ has $\mathcal{C}(v) = \emptyset$, and G is not g-choosable.

We now define the following:

$$\rho(G) = \min_{v \in V(G)} \{\chi_{\text{SC}}(G - v) + \deg(v) + 1\},$$
$$\tau(G) = \min_f \{\text{size}(f) : G \text{ is } f\text{-choosable and } 2 \leq f(v) \leq \deg(v)\}.$$

Size functions f for which $f(v) = 1$ or $f(v) > \deg(v)$ for some vertex v we call *simple size functions*, and all others, *non-simple size functions*. The following lemma is the simplest special case of Lemmas 7 and 8 in [5].

Lemma 1. *Let f be a size function on a graph G.*

(a) If $f(v) = 1$ for some vertex $v \in V(G)$, then G is f-choosable if and only if $G - v$ is $\widetilde{f}_v$-choosable.

(b) If $f(v) > \deg(v)$ for some vertex v, then G is f-choosable if and only if $G - v$ is f_{G-v}-choosable.

Lemma 2. *Let G be a graph. Then $\chi_{\text{SC}}(G) = \min\{\rho(G), \tau(G)\}$. In particular, if $G - v$ is sc-greedy for every $v \in V(G)$, then $\chi_{\text{SC}}(G) = \min\{GB(G), \tau(G)\}$.*

Proof. Let f be a simple choice function (i.e., G is f-choosable). Suppose first that $f(v) = 1$ for some $v \in V(G)$. Then $G - v$ is $\widetilde{f}_v$-choosable by Lemma 1, and we have

$$\text{size}(f) = \text{size}(\widetilde{f}_v) + \deg(v) + 1 \geq \chi_{\text{SC}}(G - v) + \deg(v) + 1 = \rho(G).$$

Secondly, suppose that $f(v) > \deg(v)$ for some $v \in V(G)$. Then then $G - v$ is f_{G-v}-choosable by Lemma 1, and since $\text{size}(f) = \text{size}(f_{G-v}) + f(v)$, we have

$$\text{size}(f) \geq \text{size}(f_{G-v}) + \deg(v) + 1 \geq \chi_{\text{SC}}(G - v) + \deg(v) + 1 = \rho(G).$$

So size$(f) \geq \rho(G)$ for any simple choice function f, and the result follows. $\square$ $\square$

The preceding lemma allows for considerable simplification of many proofs. Simple size functions can be thought of as somewhat trivial, though bothersome cases which need to be considered, and the lemma above is our attempt to dispense with much of the trouble.

Lemma 3. *Let G be a graph decomposable into blocks $G_1, \ldots, G_k$. Then*

$$\chi_{\mathrm{SC}}(G) = \sum_{j=1}^{k} \chi_{\mathrm{SC}}(G_j) - k + 1.$$

In particular, a graph all of whose blocks are sc-greedy is sc-greedy.

The lemma above follows immediately from Theorem 1 in [1]. An easy corollary is that a graph obtained from a sc-greedy graph by attaching a pendant vertex is also sc-greedy. In fact, if G' is obtained in this way from G, then $\chi_{\mathrm{SC}}(G') = \chi_{\mathrm{SC}}(G) + 2$.

2 Strings of Cycles

The following theorem answers a question raised in [1]. The symbol $\square$ denotes the Cartesian product, and P_n is the path on n vertices.

Theorem 4. *The graph $P_2 \square P_n$ is sc-greedy; that is, $\chi_{\mathrm{SC}}(P_2 \square P_n) = 5n - 2$.*

Proof. Label the vertices as in Figure 1. For any $k = 1, \ldots, n$, let G_k be the subgraph induced by vertices t_k and b_k, let L_k be the subgraph induced by vertices $t_1, b_1, \ldots, t_k, b_k$, and let R_k be the subgraph induced by vertices $t_k, b_k, \ldots, t_n, b_n$.

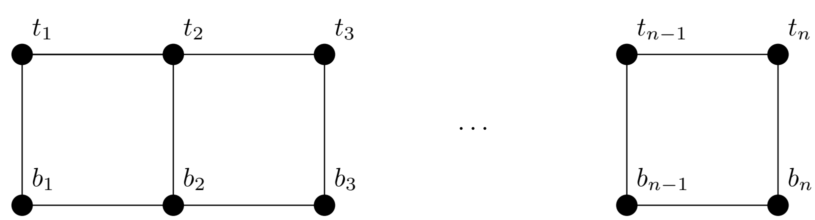

Figure 1: $P_2 \square P_n$

The proof is by induction on n. The basis P_2 is a complete graph, hence is sc-greedy. Now assume that $P_2 \square P_k$ is sc-greedy for $k < n$. We will show $G = P_2 \square P_n$ is sc-greedy; that is, its sum choice number is $5n - 2$. Let f be a function on G of size $5n - 3$. We will suppose that G is f-choosable and show that this implies that size(f) must in fact be at least $5n-2$, and hence $P_2 \,\square\, P_n$ cannot be f-choosable if size$(f) = 5n-3$.

First, it is easy to see that if size$(f_{G_k}) \leq 2$ for any $k = 1, \ldots, n$, then G is not f-choosable, so we may assume that size$(f_{G_k}) \geq 3$. Now suppose for some $1 < k < n$ that size$(f_{G_k}) \leq 4$. By the induction hypothesis we have $\chi_{\mathrm{SC}}(L_k) = 5k-2$, and hence size$(f_{L_{k-1}}) \geq 5k-6$, and similarly by the induction hypothesis $\chi_{\mathrm{SC}}(R_k) = 5(n-k+1)-2$, so size$(f_{R_{k+1}}) \geq 5(n-k)-1$. Thus we must have size$(f) \geq (5k-6)+4+(5(n-k)-1) = 5n-3$, and hence the above inequalities must be equalities. We will now define an uncolorable f-assignment $\mathcal{C}$. It is easy to check that since size$(f_{G_k}) \leq 4$, there exists an f_{G_k}-assignment $\mathcal{C}'$ such that there are at most two distinct proper $\mathcal{C}'$-colorings, c_1 and c_2, of G_k. Let $c_1 = c_2$ if there is only one. Let g_1 be the size function on L_{k-1} defined by $g_1(v) = f(v) - 1$ if $v \in V(G_{k-1})$ and $g_1(v) = f(v)$ otherwise, and let g_2 be the size function on R_{k+1} defined by $g_2(v) = f(v) - 1$ if $v \in V(G_{k+1})$ and $g_2(v) = f(v)$ otherwise. As size$(g_1) < \chi_{\mathrm{SC}}(L_{k-1})$ and size$(g_2) < \chi_{\mathrm{SC}}(R_{k+1})$, L_{k-1} is not g_1-choosable, and R_{k+1} is not g_2-choosable. Hence there exists a g_1-assignment $\mathcal{C}_1$ and a g_2-assignment $\mathcal{C}_2$, neither of which has a proper coloring. Moreover, we may name the colors so that $\mathcal{C}'(G_k)$ is disjoint from $\mathcal{C}_1(L_{k-1})$ and $\mathcal{C}_2(R_{k+1})$. Define $\mathcal{C}$ by $\mathcal{C} = \mathcal{C}'$ on G_k, $\mathcal{C} = \mathcal{C}_1$ on L_{k-1}, and $\mathcal{C} = \mathcal{C}_2$ on R_{k+1}, except that we append $c_1(t_k)$, $c_1(b_k)$, $c_2(t_k)$, and $c_2(b_k)$ to $\mathcal{C}_1(t_{k-1})$, $\mathcal{C}_1(b_{k-1})$, $\mathcal{C}_2(t_{k+1})$, and $\mathcal{C}_2(b_{k+1})$, respectively. Let c be a $\mathcal{C}$-coloring. If c is proper, then c_{G_k} is equal to either c_1 or c_2. If $c_{G_k} = c_1$, then $c_{L_{k-1}}$ must be a proper $\mathcal{C}_1$-coloring of L_{k-1}, and if $c_{G_k} = c_2$, then $c_{R_{k-1}}$ must be a proper $\mathcal{C}_2$-coloring of R_{k-1}, neither of which exists. Hence size$(f_{G_k}) \geq 5$ for $k = 2, \ldots, n-1$.

If $f(t_1) = 1$, then by Lemma 1, G is f-choosable if and only if $G - t_1$ is $\widetilde{f}_{t_1}$-choosable. However, $G - t_1$ is sc-greedy by the induction hypothesis and the comment following Lemma 3, hence $\chi_{\mathrm{SC}}(G - t_1) = 5n-5 > 5n-6 = \mathrm{size}(\widetilde{f}_{t_1})$, so G is not f-choosable. A similar argument applies if any of b_1, t_n, or b_n has list size 1. Thus size$(f_{G_1}) \geq 4$ and size$(f_{G_n}) \geq 4$, and hence size$(f) = \sum_{k=1}^{n} \mathrm{size}(f_{G_k}) \geq 5(n-2) + 2(4) = 5n-2$. $\square$ $\square$

In addition to the above result, we have determined by a lengthy case analysis that $P_3 \,\square\, P_n$ has sum choice number GB $-\lfloor \frac{n-1}{8} \rfloor$. See [3] for details. Moreover, ideas very similar to those used in the proof above could likely be used to show that if instead of all 4-cycles, we used cycles of arbitrary and varying lengths greater than 3, the graph obtained is still sc-greedy. In fact, if, instead of merely laying cycles edge-to-edge, we laid them along a tree structure, the resulting graph would still be sc-greedy. What other underlying structures lead to sc-greedy graphs? We leave these problems for interested readers.

However, 3-cycles are somewhat more complicated to deal with. Consider the graph pictured in Figure 2, obtained by laying $n-2$ triangles edge-to-edge. This is the graph is P_n^2. Formally, it has vertex set $v_1, \ldots, v_n$ with v_i adjacent to v_j if and only if $0 < |i-j| \leq 2$. Below we prove that it is sc-greedy. A longer proof using the same techniques can be used to show that for any choice function f of minimum size on P_n^2, there exists an f-assignment forcing the vertices v_1 and v_2 to be specific colors.

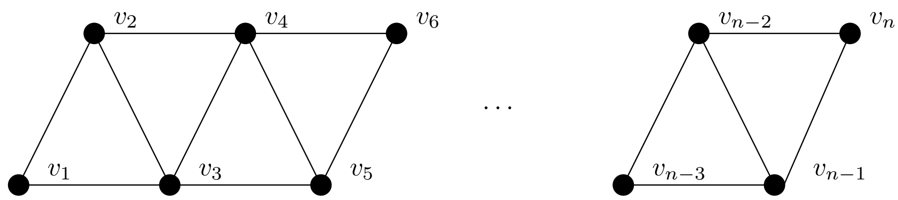

Figure 2: P_n^2

Theorem 5. *The graph P_n^2 is sc-greedy; that is, $\chi_{\mathrm{SC}}(P_n^2) = 3n - 3$.*

Proof. The proof is by induction on n. Note that P_1^2, P_2^2 and P_3^2 are complete graphs, and hence are sc-greedy. Now assume that P_m^2 is sc-greedy for all $m < n$. Note that removing a vertex from P_n^2 leaves a graph whose blocks are either paths or copies of P_m^2 for values of m less than n. By the induction hypothesis and Lemma 3, such a graph is sc-greedy. Thus by Lemma 2, it remains to show $\tau(P_n^2) \geq 3n - 3$. That is, we must now show that for any non-simple size function f of size $3n - 4$, P_n^2 is not f-choosable. Let R_k denote the subgraph of P_n^2 which is induced by the vertices $v_k, \ldots, v_n$.

Note that $f(v_1) = 2$ since $\deg(v_1) = 2$. Let j be the least index greater than 1 of a vertex having list size 2. This index must exist and be at most $n - 2$, as otherwise size(f) would exceed $3n - 4$. As $j \leq n - 2$, R_j and R_{j+1} are defined. Note that if there is a vertex v_k, $k < j$, with $f(v_k) \geq 4$, then size$(f_{R_j}) < \chi_{\mathrm{SC}}(R_j)$. Thus we may assume that $f(v_1) = f(v_j) = 2$ and $f(v_k) = 3$ for $1 < k < j$. We now create an f-assignment $\mathcal{C}$ with no proper coloring. Let $\mathcal{C}(v_1) = 12$ and $\mathcal{C}(v_i) = 123$ for $1 < i < j$. Let $\mathcal{C}(v_j)$ be 34 if j is congruent to 1 modulo 3, and $\mathcal{C}(v_j) = 12$ otherwise. We will define $\mathcal{C}$ on R_{j+1} differently according to the congruence of j modulo 3. The following can easily be checked:

$(*)$ No proper $\mathcal{C}$-coloring can use color 3 on v_i for each $i \leq j$ congruent to 1 modulo 3.

If $j \equiv 0 \pmod 3$, let g be a size function on R_j with $g(v_{j+1}) = f(v_{j+1}) - 1$, and let g agree with f elsewhere. Since size$(g) < \chi_{\mathrm{SC}}(R_j)$, R_j is not g-choosable. Let $\mathcal{C}'$ be an uncolorable g-assignment with $\mathcal{C}'(v_j) = 12$ and $3 \notin \mathcal{C}'(v_{j+1})$. Define $\mathcal{C}$ on R_{j+1} by letting $\mathcal{C}$ equal $\mathcal{C}'$, except that we append color 3 to the list for v_{j+1}. By $(*)$, any proper $\mathcal{C}$-coloring cannot use color 3 on v_{j-2}. Therefore, since $\mathcal{C}(v_j) = 12$, colors 1 and 2 must be used on v_{j-2} and v_j, leaving only color 3 to be used on v_{j-1}. Therefore, color 3 is unavailable on v_{j+1}, and we must color R_j from $\mathcal{C}'$, which is not possible.

If $j \equiv 1 \pmod 3$, let g be as in the previous paragraph, and let $\mathcal{C}'$ be an uncolorable g-assignment with $4 \notin \mathcal{C}'(v_{j+1})$. Any proper $\mathcal{C}$-coloring cannot use color 3 on v_j by $(*)$. Therefore color 4 is used on v_j, and hence is unavailable on v_{j+1}. So we must color R_j from $\mathcal{C}'$, which is not possible.

Finally, if $j \equiv 2 \pmod 3$, let g be a size function on R_{j+1} with $g(v_{j+1}) = f(v_{j+1}) - 2$, and let g agree with f elsewhere. Since size$(g) < \chi_{\mathrm{SC}}(R_{j+1})$, R_{j+1}

is not g-choosable. Let $\mathcal{C}'$ be an uncolorable g-assignment such that neither color 1 nor color 2 appears on $\mathcal{C}'(v_{j+1})$. Define $\mathcal{C}$ on R_{j+1} by letting $\mathcal{C}$ equal $\mathcal{C}'$, except that we append colors 1 and 2 to the list for v_{j+1}. By $(*)$, A proper $\mathcal{C}$-coloring must not use color 3 on v_{j-1}. Therefore, since $\mathcal{C}(v_j) = 12$, colors 1 and 2 must be used on v_{j-1} and v_j and are therefore unavailable to be used on v_{j+1}. So we must color R_{j+1} from $\mathcal{C}'$, which is not possible. □ □

3 Theta Graphs and the Petersen Graph

The authors of [1] asked about the choice number of the Petersen graph. We will need the following result. By a theta graph, θ_{k_1,k_2,k_3}, we mean a simple graph consisting of two vertices connected by three internally vertex disjoint paths, having k_1, k_2, and k_3 internal vertices, respectively, $0 \leq k_1 \leq k_2 \leq k_3$. Recall that we denote the greedy bound, the sum of the number of vertices and edges, by GB, which in this case is $2(k_1 + k_2 + k_3) + 5$.

Theorem 6. $\chi_{\mathrm{SC}}(\theta_{k_1,k_2,k_3}) = \begin{cases} \mathrm{GB}-1, & \textit{if } k_1 = k_2 = 1 \textit{ and } k_3 \textit{ is odd} \\ \mathrm{GB}, & \textit{otherwise.} \end{cases}$

Proof. Removing a vertex from a theta graph leaves either a tree or a cycle with pendant paths, both of which are sc-greedy. Hence by Lemma 2, it remains to determine $\tau(\theta_{k_1,k_2,k_3})$. If f is a non-simple size function with $\mathrm{size}(f) = \mathrm{GB}-1 = 2(k_1 + k_2 + k_3) + 4$, then $f \equiv 2$, since the vertex set has size $k_1 + k_2 + k_3 + 2$. However, by a well-known result in [2], the only theta graphs which are 2-choosable have $k_1 = k_2 = 1$ and k_3 odd. □ □

To show a graph G is sc-greedy, by Lemma 2 it suffices to show for any vertex v of G that $G - v$ is sc-greedy, and then show that there is no non-simple choice function of size one less than the greedy bound. The same of course applies to $G - v$, and so we get a recursive procedure whereby we remove vertices from G until we get to graphs we know are sc-greedy, and at each stage we show that there are no non-simple choice functions of size one less than the greedy bound.

The following will be important in the proof below: Odd cycles are not 2-choosable, because the list assignment with all lists equal to 12 has no proper coloring. Moreover, this implies that if we assign lists 12 to all vertices of an odd cycle but one, which gets list 123, then color 3 must be used on that vertex.

Theorem 7. *The Petersen Graph is sc-greedy; that is, it has sum choice number 25.*

Proof. Denote the Petersen graph by P, let Q denote P minus a vertex, and let R denote Q minus a vertex of degree 2. The greedy bound is 25 for P, 21 for Q, and 18 for R.

Recall that appending a vertex to a sc-greedy graph produces a sc-greedy graph by the comment following Lemma 3. For any vertex $v \in V(R)$, $R - v$ is either a sc-greedy theta graph or a cycle with pendant paths, and hence is sc-greedy. Moreover, for any vertex $v \in V(Q)$ of degree 3, $Q - v$ is a sc-greedy

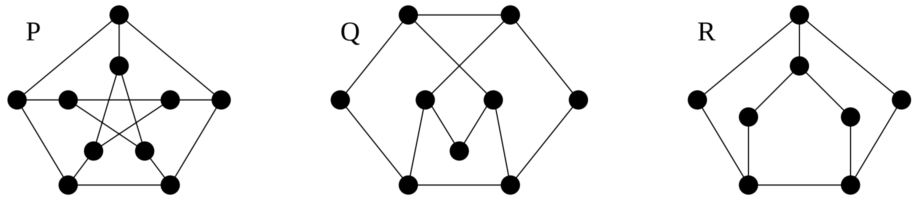

Figure 3: The graphs P, Q, and R

theta graph with pendant edges. Thus it remains to consider non-simple size functions of size one less than the greedy bound on each of P, Q, and R.

The only non-simple size functions of size 17 on R assign list size 2 for all but one vertex, and hence there is a 5-cycle all of whose list sizes are 2, which is not colorable. The only non-simple size functions of size 20 on Q assign list size 2 to all but two vertices v and w, both having degree 3. It can be checked that there is a 5-cycle avoiding any pair of adjacent vertices, and hence if v and w are adjacent, then there is a 5-cycle all of whose list sizes are 2, which is not colorable. If, on the other hand, v and w are not adjacent, then they must be at distance two from each other. Let x denote their common neighbor. It can be checked that there exist 5-cycles C_1 and C_2 with v in C_1, but not C_2, w in C_2, but not C_1, and x not in either. Let f be a non-simple size function of size 20, and create an f-assignment $\mathcal{C}$ with $\mathcal{C}(v) = 123$, $\mathcal{C}(w) = 124$, $\mathcal{C}(x) = 34$, and let any other vertex have list 12. These lists force color 3 on v and color 4 on w. Hence there is no proper $\mathcal{C}$-coloring because $\mathcal{C}(x) = 34$.

Thus it remains to consider non-simple size functions of size 24 on P. Any such size function assigns list size 3 to four vertices and list size 2 to all others. Let H denote the subgraph induced by the vertices assigned list size 3. We consider cases according to the possibilities for H.

If H is a path, P_4, then it can be checked that there exists a 5-cycle all of whose list sizes are 2, which is not colorable. The other possibilities are shown in Figure 4 along with uncolorable list assignments. The vertices of H are indicated with solid circles, and vertices with no list specified can have any list. Note that by symmetry, these pictures give the only layouts of H that need be considered. It is straightforward for the reader to check that these assignments are uncolorable. $\square$ $\square$

4 Complete bipartite graphs

Let f and G be given, and let $\mathcal{C}$ be an f-assignment. Let Y be an independent set in G, and let X be the subgraph of G induced by the vertices not in Y. Let c be a proper $\mathcal{C}_X$-coloring. For any $y \in Y$, $c[N(y)]$ is the set of colors used by c on $N(y)$, the neighborhood of y. We will say c is *blocked* by Y if there is a

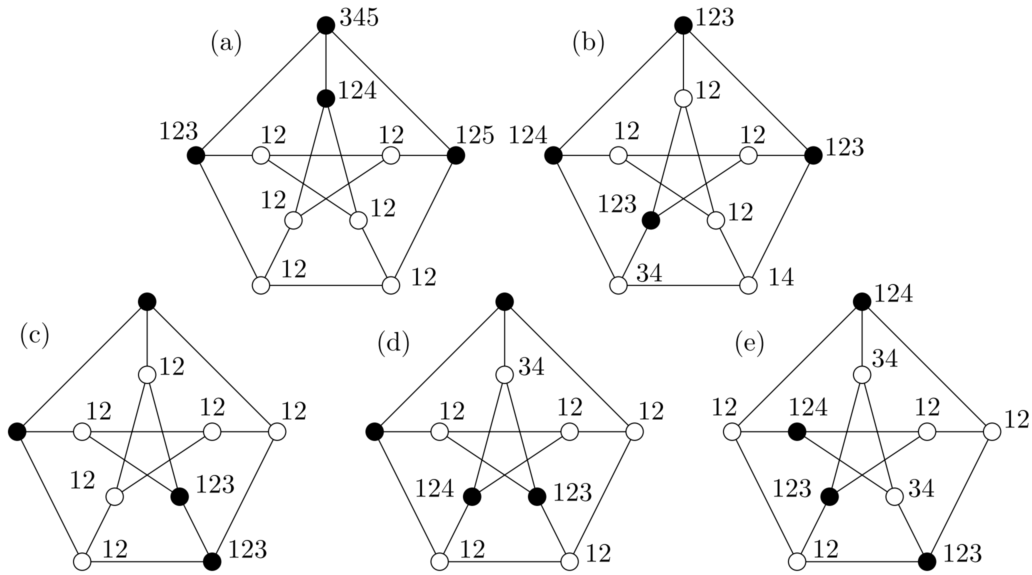

Figure 4: Uncolorable list assignments for Theorem 7

vertex $y \in Y$ such that $\mathcal{C}(y) \subset c[N(y)]$. In other words, c cannot be extended to a proper $\mathcal{C}$-coloring of all of G. We then have the following.

Lemma 8. *Let G, f, $\mathcal{C}$, X, and Y be as above. Then G is $\mathcal{C}$-colorable if and only if there exists a proper $\mathcal{C}_X$-coloring which is not blocked by Y.*

The proof is straightforward. It then follows from the definition of choosability that G is f-choosable if and only if for every f-assignment $\mathcal{C}$, some proper $\mathcal{C}_X$-coloring is not blocked by Y.

We can apply these ideas to the computation of the sum choice number of the complete bipartite graph $K_{p,q}$, $p \leq q$. Let X and Y denote the partite sets of size p and q, respectively, with $X = \{x_1, \ldots, x_p\}$. Let $\alpha(p, q)$ be the minimum size of a choice function f with $f(Y) = \{2\}$, let $\beta(p, q)$ be the minimum size of a choice function f with $f(Y) \subset \{2, \ldots, p\}$, and let $\gamma(p, q)$ be the minimum size over all other choice functions f. Clearly, $\chi_{\mathrm{SC}}(K_{p,q})$ is given by the minimum of these three values. By the ideas in the proof of Lemma 2, $\gamma(p, q) = \chi_{\mathrm{SC}}(K_{p,q-1}) + p + 1$. We will use the blocking idea to compute $\alpha(p, q)$ for a fixed p. Let f be a size function on $K_{p,q}$ with $f(Y) = \{2\}$, and let $\mathcal{C}$ be an f-assignment. Since X is an independent set, the collection of all proper $\mathcal{C}_X$-colorings can be identified with all p-tuples $(a_1, \ldots, a_p)$, $a_i \in \mathcal{C}(x_i)$ for each $i = 1, \ldots, p$. Since $N(y) = X$ for every $y \in Y$, the set $c[N(y)]$ becomes $c[X]$ for each y, and a proper $\mathcal{C}_X$-coloring c is blocked if and only if $\mathcal{C}(y) \subset c[X]$ for some $y \in Y$.

The sum choice number of $K_{2,q}$ was determined in [1]. We provide a somewhat similar proof here which will generalize to $K_{3,q}$.

Theorem 9 (Berliner et al.). *The sum choice number of $K_{2,q}$ is given by*

$$\chi_{\text{SC}}(K_{2,q}) = 2q + \min\{l + m : q < lm, \text{ with } l, m \in \mathbb{N}\}.$$

Proof. We will compute $\alpha(2, q)$ and then show $\alpha(2, q) \leq \gamma(2, q)$. Note that $\alpha(2, q) = \beta(2, q)$. Fix positive integers l and m. Consider a size function f on $K_{2,q}$ with $f(x_1) = l$, $f(x_2) = m$, and $f(Y) = \{2\}$. Using the blocking idea, if $\mathcal{C}$ is an f-assignment such that there exists a color a in $\mathcal{C}(x_1) \cap \mathcal{C}(x_2)$, then we get a proper $\mathcal{C}$-coloring by coloring x_1 and x_2 with a, since there can be no 2-set contained in $\{a\}$. If the lists on X are disjoint, then there are a total of lm proper colorings from the lists on X, and each vertex of Y can be used to block exactly one of them. Thus if $q < lm$, there is always some proper coloring not blocked, whereas if $q \geq lm$, then there exists a list assignment blocking every proper coloring. We conclude that $\alpha(2, q) = 2q + \min\{l + m : q < lm, \text{ with } l, m \in \mathbb{N}\}$.

We will now show $\alpha(2, q) \leq \gamma(p, q) = \chi_{\text{SC}}(K_{2,q-1}) + 3$ by induction. For the base case, $\alpha(2, 1) = 5 = \chi_{\text{SC}}(K_{2,0}) + 3$. Now assume the inequality holds for $q - 1$. Then $\chi_{\text{SC}}(K_{2,q-1}) = \alpha(2, q - 1)$. Hence the inequality for q holds if and only if $M \leq N+1$ where $M = \min\{l+m : q < lm\}$ and $N = \min\{l+m : q-1 < lm\}$, with both minima taken over positive integers. Pick (l^*, m^*) giving the minimum, N. Then $q < l^*m^*+1 \leq l^*(m^*+1)$. Hence $M \leq l^*+(m^*+1) = N+1$.
□ □

The proof above and Lemma 1 combine to give a characterization of choosability for $K_{2,q}$. Let $u = |f^{-1}(1) \cap V(Y)|$ and $d = |f^{-1}(2) \cap V(Y)|$. It is straightforward to show that $K_{2,q}$ is f-choosable if and only if $d < (f(x_1)-u)(f(x_2)-u)$.

Corollary 10. *Explicitly, the sum choice number of $K_{2,q}$ is given by*

$$\chi_{\text{SC}}(K_{2,q}) = 2q + 1 + \lfloor\sqrt{4q + 1}\rfloor.$$

Proof. We compute $\alpha(2, q)$ explicitly. Suppose $(l, m) = (l, l+t)$ for $t > 1$. Then if $q < lm$, we have $q < l(l + t) \leq l^2 + lt + t - 1 = (l + 1)(l + t - 1)$. Hence the minimum, $\alpha(2, q)$, must occur at (l, m) of the form (l, l) or $(l, l + 1)$. Suppose (l^*, m^*) gives the minimum. Then l^* must satisfy $(l^* - 1)l^* \leq q < l^*(l^* + 1)$ and m^* must satisfy $(m^* - 1)^2 \leq q < (m^*)^2$. The second inequality is equivalent to $m^* - 1 \leq \sqrt{q} < m^*$, and so $m^* = \lfloor\sqrt{q}\rfloor + 1$. To find a similar expression for l^*, let $g(x) = (\sqrt{4x + 1} - 1)/2$. This function is increasing and satisfies $g(x(x+1)) = x$ for $x \geq 0$. Applying it to the first inequality gives $l^* - 1 \leq g(q) < l^*$. Therefore, $l^* = \lfloor g(q)\rfloor + 1$.

Thus $\chi_{\text{SC}}(K_{2,q}) = 2q + 2 + \lfloor\sqrt{q}\rfloor + \lfloor g(q)\rfloor$, and this quantity is equal to $2q + 1 + \lfloor\sqrt{4q + 1}\rfloor$. To see this, let $r = \lfloor\sqrt{4q + 1}\rfloor$ and $s = \lfloor\sqrt{4q}\rfloor$. If $r = s$ it is easy to check that the two quantities are equal by considering r odd and even separately. If $r = s + 1$, then $4q + 1$ is an odd perfect square, hence we need only check that the two quantities are equal for odd r, which is easily seen to be true. □ □

These same techniques can be used to find the sum choice number of $K_{3,q}$. Let f be a size function on $K_{3,q}$ satisfying $f(Y) \subset \{2, 3\}$, $f(x_1) = l$, $f(x_2) = m$, and $f(x_3) = n$, with $0 < l \leq m \leq n$, and let $t = |f^{-1}(3) \cap V(Y)|$. We will denote this by $f = (l, m, n : t)_q$. When we use this notation it will be implicit that $f(Y) \subset \{2, 3\}$.

We provide an example here to motivate the proof of Theorem 11. In the proof of Theorem 9, we considered any size function f satisfying $f(x_1) = l$, $f(x_2) = m$ and $f(Y) = \{2\}$. For the sake of illustration, suppose that $l = 2$ and $m = 3$. The only f-assignment of interest has disjoint lists on x_1 and x_2, say $\mathcal{C}(x_1) = 12$ and $\mathcal{C}(x_1) = 345$. We could be certain that every proper coloring is blocked provided we assign the lists 13, 14, 15, 23, 24, 25 on Y. If any of these lists is missing, then there exists a proper $\mathcal{C}$-coloring. Hence we conclude that $K_{2,q}$ is f-choosable if and only if $q < 6$.

For $K_{3,q}$ things are complicated by the fact that there are now list assignments of interest where the lists are not all disjoint. Consider the size function $(4, 4, 4 : 0)_q$. If we put lists 1234, 5678, and *abcd* on the vertices of X, it turns out that 16 2-sets is the minimum number needed to block every proper coloring of X, namely all 2-sets with one element coming from $\{1, 2, 3, 4\}$ and the other from $\{5, 6, 7, 8\}$. If instead we put lists 1234, 1256, and 3456 on X, then only 12 2-sets are needed to block every proper coloring, namely 13, 14, 15, 16, 23, 24, 25, 26, 35, 36, 45, and 46. It turns out that these X-lists are worst possible in the sense that they require the least number of 2-sets to block every proper coloring. That is, regardless of the collection of size 4 lists we put on X, if there are less than 12 vertices in Y, then there is always a proper coloring of the entire graph. Hence we conclude that $K_{3,q}$ is $(4, 4, 4 : 0)_q$-choosable if and only $q < 12$. By finding the worst possible lists for any $l \leq m \leq n$ we get a quantity, $q^*(l, m, n)$, which gives the minimum value of q such that $K_{3,q}$ is not $(l, m, n : 0)_q$-choosable. Thus we conclude that $\alpha(3, q) = 2q + \min\{l + m + n : q < q^*(l, m, n)\}$, with the minimum taken over $l, m, n \in \mathbb{N}$. Certain properties of $q^*(l, m, n)$ will allow us to show that $\alpha(3, q) \leq \beta(3, q)$, and a similar argument to the one used in Theorem 9 will show $\alpha(3, q) \leq \gamma(3, q)$.

Theorem 11. *The sum choice number of $K_{3,q}$ is given by*

$$2q + \min\{l + m + n : q < q^*(l, m, n), \text{ with } l, m, n \in \mathbb{N},\ l \leq m \leq n\},$$

where $q^*(l, m, n)$ is given by $lm - \lfloor (l + m - n)^2/4 \rfloor$ if $n \leq l + m$, and by lm otherwise.

Proof. We will first compute $\alpha(3, q)$, then show $\beta(3, q) = \alpha(3, q)$, and finally show that $\alpha(3, q) \leq \gamma(3, q)$. Fix positive integers l,m, and n, with $l \leq m \leq n$. Consider the size function $f = (l, m, n : 0)_q$, and let $\mathcal{C}$ be an f-assignment. We will determine the minimum number of 2-sets needed to block every proper $\mathcal{C}_X$-coloring. If there exists a color a in $\mathcal{C}(x_1) \cap \mathcal{C}(x_2) \cap \mathcal{C}(x_3)$, then we get a proper $\mathcal{C}$-coloring by coloring x_1, x_2, and x_3 with a, since there can be no 2-set contained in $\{a\}$. So assume there is no color in common to all the lists on X. In this case, $\mathcal{C}_X$ has the following form:

$$\begin{aligned}
\mathcal{C}(x_1) &= a_1 \dots a_{k_1}\, b_1 \dots b_{k_2}\, c_1 \dots c_{k_4}, \\
\mathcal{C}(x_2) &= a_1 \dots a_{k_1}\, d_1 \dots d_{k_3}\, e_1 \dots e_{k_5}, \\
\mathcal{C}(x_3) &= b_1 \dots b_{k_2}\, d_1 \dots d_{k_3}\, f_1 \dots f_{k_6}.
\end{aligned}$$

Colors with different names are distinct, and some of the k_i may be zero. In order to block each proper coloring, we require all 2-sets of the forms a_ib_j, a_id_j, a_if_j, b_id_j, b_ie_j, c_id_j, where i and j range over all possible values. The sets remaining unblocked are of the form $\{c_i, e_j, f_k\}$. To block these with the minimum number of 2-sets, add to the collection all 2-sets of the forms c_ie_j, c_if_j, or e_if_j, whichever gives the least number. In total, a smallest collection of 2-sets must have $k_1n + k_2(m-k_1) + k_3(l-k_1-k_2) + \min\{(l-k_1-k_2)(m-k_1-k_3), (l-k_1-k_2)(n-k_2-k_3), (m-k_1-k_3)(n-k_2-k_3)\}$, which simplifies to $\min\{\delta(l,m,n,k_1), \delta(l,n,m,k_2), \delta(m,n,l,k_3)\}$, where $\delta(x,y,z,w) = xy + w(z - x - y + w)$.

We now minimize this over all possible $\mathcal{C}_X$-assignments to find the list assignment requiring the least number of 2-sets to block every proper coloring. The minimum number of 2-sets needed in this case will be denoted by $q^*(l,m,n)$, which in fact gives the minimum value of q such that $K_{3,q}$ is not $(l,m,n:0)_q$-choosable. To determine a formula for $q^*(l,m,n)$, we determine the minimum of the expression in the previous paragraph over all nonnegative integers k_1, k_2, and k_3 satisfying $l \geq k_1, k_2 \geq 0$ and $m \geq k_3 \geq 0$. Note that $\delta(l,m,n,k_1)$ is a quadratic function in k_1, and a simple analysis shows that the minimum occurs at $k_1 = \lfloor (l+m-n)/2 \rfloor$ for $n \leq l+m$ and $k_1 = 0$ for $n > l+m$. A similar analysis applies to the other two delta quantities, and it can be checked that the minimum obtained from each of the three delta quantities is equal to $lm - \lfloor (l+m-n)^2/4 \rfloor$ for $n \leq l+m$ and lm for $n > l+m$. This quantity is $q^*(l,m,n)$. To summarize, with q^* as mentioned, we have

$$\alpha(3,q) = 2q + \min\{l+m+n : q < q^*(l,m,n),\ \text{with } l,m,n \in \mathbb{N},\ l \leq m \leq n\}.$$

Now we show $\beta(3,q) = \alpha(3,q)$. Note that this is clearly true when $q = 0$. For $q > 0$ we show that $K_{3,q}$ is not g-choosable for any function $g = (l_0, m_0, n_0 : t)_q$ of size $\alpha(3,q) - 1$ with $t > 0$. We will assume on the contrary that $K_{3,q}$ is g-choosable, and construct a sequence of size functions $h_i = (l_i, m_i, n_i : t-i)_q$ for $i = 0, \ldots, t$ (with $h_0 = g$), such that if $K_{3,q}$ is h_i-choosable, then it is also h_{i+1}-choosable, and then show that $K_{3,q}$ is in fact not h_t-choosable, thereby contradicting our assumption that it is g-choosable. Let $d_i = |h_i^{-1}(2) \cap Y|$. We may assume that $n_0 > 1$ as otherwise g must equal $(1,1,1:t)_q$, and $K_{3,q}$ is not $(1,1,1:t)_q$-choosable for $q > 0$ and any t. Let $i_0 = n_0 - m_0$ provided $t \geq n_0 - m_0$, and otherwise let $i_0 = t$. For $i = 1, \ldots i_0$, let $l_i = l_0$, $m_i = m_{i-1} + 1$, and $n_i = n_0$. Let $i_1 = i_0 + 1$ if $l_0 = 1$, and let $i_1 = i_0$ if $l_0 > 1$. If $l_0 = 1$, let $l_{i_1} = 2$, $m_{i_1} = m_{i_0}$, and $n_{i_1} = n_{i_0}$. For $j \geq 1$, let $l_{i_1+j} = l_{i_1}$, $m_{i_1+j} = m_{i_1} + \lfloor j/2 \rfloor$ and $n_{i_1+j} = n_{i_1} + \lceil j/2 \rceil$.

Now, for $l,m,n \in \mathbb{N}$ one can easily compute that $q^*(l,m,n)$ is strictly greater than both $q^*(l-1,m,n)$ and $q^*(l,m-1,n)$, and if $n < l+m$, then $q^*(l,m,n)$ is strictly greater than $q^*(l,m,n-1)$. Note that we have arranged it so that for each $i = 1, \ldots, t$, $l_i \leq m_i \leq n_i$, and for $i = i_1, \ldots, t$, $n_i < l_i + m_i$. Let $q_i = q^*(l_i, m_i, n_i)$. By assumption, $K_{3,q}$ is h_0-choosable. Now let $0 \leq i < t$ and assume $K_{3,q}$ is h_i-choosable. Then $d_i < q_i$. Thus we have $d_{i+1} = d_i + 1 < q_i + 1 \leq q_{i+1}$, by the strict monotonicity of q^* in each argument. Hence $K_{3,q}$

is also h_{i+1}-choosable. Thus $K_{3,q}$ is h_t-choosable and of size $\alpha(3,q)-1$. Since $h_t^{-1}(3)=\emptyset$ this contradicts the definition of $\alpha(3,q)$, and so we have obtained our contradiction.

Finally, we show by induction that $\alpha(3,q) \le \gamma(3,q) = \chi_{\mathrm{SC}}(K_{3,q-1})+4$. For the base case, $\alpha(3,1) = 7 = \chi_{\mathrm{SC}}(K_{3,0})+4$. Assume the inequality holds for $q-1$. Then $\chi_{\mathrm{SC}}(K_{3,q-1}) = \alpha(3,q-1)$. Hence the inequality for q holds if and only if $M \le N+2$ where $M = \min\{l+m+n : q < q^*(l,m,n)\}$ and $N = \min\{l+m+n : q-1 < q^*(l,m,n)\}$, with both minimums taken over positive integers. Pick (l^*,m^*,n^*) giving the minimum, N. Then $q < l^*m^* - \lfloor\frac{(l^*+m^*-n^*)^2}{4}\rfloor + 1 \le l^*(m^*+1) - \lfloor\frac{(l^*+(m^*+1)-(n^*+1))^2}{4}\rfloor$. Hence $M \le l^*+(m^*+1)+(n^*+1) = N+2$. □ □

Corollary 12. *Explicitly, the sum choice number of $K_{3,q}$ is given by*

$$\chi_{\mathrm{SC}}(K_{3,q}) = 2q+1+\lfloor\sqrt{12q+4}\rfloor.$$

Proof. We compute $\alpha(3,q)$ explicitly. We first show that to determine the minimum $\alpha(3,q)$ it suffices to consider only those (l,m,n) of the form (l,l,l), $(l,l,l+1)$, and $(l,l+1,l+1)$. Note that by definition, $q^*(l,m,n) = lm$ for $n \ge l+m+1$, and since $q^*(l,m,l+m-1) = lm$ as well, we only need to consider $n < l+m$. That is, for all (l,m,n) that need to be considered, we have $q^*(l,m,n) = lm - \lfloor(l+m-n)^2/4\rfloor$. Suppose first that (l,m,n) is such that $n-l > 1$ and l, m and n are all distinct. Then

$$\begin{aligned} q^*(l+1,m,n-1) &= (l+1)m - \lfloor(l+m-n+2)^2/4\rfloor \\ &= (l+1)m - \lfloor(l+m-n)^2/4\rfloor - (l+m-n) - 1 \\ &= q^*(l,m,n) + n + l - 1 \\ &> q^*(l,m,n). \end{aligned}$$

Similar computations apply if l, m, and n, are not distinct. Namely $q^*(l,l,m) < q^*(l,l+1,n-1)$, and $q^*(l,m,m) < q^*(l+1,m-1,m)$. We conclude that $\alpha(3,q)$ is minimized by (l,m,n) satisfying $n-l \le 1$.

Let (l^*,m^*,n^*) give the minimum. Define functions $g_1(x) = (\sqrt{12x+4}-2)/3$, $g_2(x) = (\sqrt{12x+4}-1)/3$, and $g_3(x) = \sqrt{12x}/3$. Note that these functions are increasing for $x \ge 0$ and satisfy $g_1(q^*(l,l+1,l+1)) = g_2(q^*(l,l,l+1)) = g_3(q^*(l,l,l)) = l$ for all $l \in \mathbb{N}$. Now (l^*,m^*,n^*) must satisfy

$$\begin{aligned} &q^*(l^*-1,l^*,l^*) \le q < q^*(l^*,l^*+1,l^*+1), \\ &q^*(m^*-1,m^*-1,m^*) \le q < q^*(m^*,m^*,m^*+1), \\ &q^*(n^*-1,n^*-1,n^*-1) \le q < q^*(n^*,n^*,n^*). \end{aligned}$$

Applying g_1 to the first inequality, g_2 to the second, and g_3 to the third gives $l^* = \lfloor g_1(q)\rfloor + 1$, $m^* = \lfloor g_2(q)\rfloor + 1$, and $n^* = \lfloor g_3(q)\rfloor + 1$.

Thus $\chi_{\mathrm{SC}}(K_{3,q}) = 2q+3+g_1(q)+g_2(q)+g_3(q)$, and this quantity is equal to $2q+1+\lfloor\sqrt{12q+4}\rfloor$. To see this, let $r = \lfloor\sqrt{12q+4}\rfloor$ and $s = \lfloor\sqrt{12q}\rfloor$. If $r = s$ it is easy to check that the two quantities are equal by considering the cases $r \equiv 0, 1, 2$ modulo 3 separately. If $r = s + 1$, then $12q + 1$ is a perfect square not divisible by 3, hence we need only check that the two quantities are equal only for $r \equiv 1, 2$ modulo 3, which is easily seen to be true. $\square$ $\square$

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