Fractions

A lot of people find working with fractions tricky. Here we go over the basic concepts and arithmetic of fractions.

First, fractions always stand for division. For instance, the fraction $\frac{12}{3}$ stands for $12 \div 3$.

Fractions and zero

There are two main facts:

- Dividing by 0 is not allowed, so 0 can never be in the denominator of a fraction. For instance, $\frac{2}{0}$ and $\frac{0}{0}$ are both undefined.
- If the numerator is 0, then the overall fraction equals 0. For instance, $\frac{0}{2}$ and $\frac{0}{17}$ are both 0. The one exception is that $\frac{0}{0}$ is undefined.

Multiplying fractions

Multiplying fractions is the easiest operation. Just multiply top and bottom separately. For instance,

$$\frac{2}{3} \cdot \frac{4}{7} = \frac{8}{21}.$$

Dividing fractions

The key to dividing fractions is to flip the denominator and multiply both numerator and denominator by it. Here are some examples.

Example 1 Compute $\dfrac{\frac{2}{3}}{\frac{4}{7}}$.

Solution: Flip the denominator to get $\frac{7}{4}$ and multiply top and bottom by it to get

$$\frac{\frac{2}{3}}{\frac{4}{7}} \cdot \frac{\frac{7}{4}}{\frac{7}{4}} = \frac{\frac{14}{12}}{\frac{28}{28}} = \frac{14}{12}.$$

The key thing to notice is that when we multiply the denominator by its flipped version (called its *reciprocal*, the result always simplifies to 1.

Example 2 Compute $\dfrac{\frac{1}{2}}{\frac{9}{4}}$.

Solution: Flip the denominator to get $\frac{4}{9}$ and multiply top and bottom by it to get

$$\frac{\frac{1}{2}}{\frac{9}{4}} \cdot \frac{\frac{4}{9}}{\frac{4}{9}} = \frac{4}{18}.$$

Adding and subtracting

Adding and subtracting fractions is a little trickier than multiplication. We can't just add tops and bottoms separately as if we tried that with say $\frac{1}{2} + \frac{1}{2}$, we would get $\frac{2}{4}$ as an answer, instead of the correct answer, which is 1.

The key idea is to find what's called a *common denominator*. People are often preoccupied with finding the least common denominator, but usually any common denominator will work.

One way that always works is to multiply each fraction by the other fraction's denominator. For example, suppose we have $\frac{2}{3} + \frac{5}{7}$. We do the following:

$$\frac{2}{3} \cdot \frac{7}{7} + \frac{5}{7} \cdot \frac{3}{3}$$

This then becomes

$$\frac{14}{21} + \frac{15}{21}.$$

The common denominator is 21. Once two fractions have the same denominator, add their numerators together, keeping the same denominator, to get $\frac{29}{21}$, which is the answer.

The reason that we need the denominators to be the same is that we want to be adding like things. Adding $\frac{2}{3}$ and $\frac{5}{7}$ is sort of like combining apples and oranges, while adding $\frac{14}{21}$ and $\frac{15}{21}$ is combining like objects.

Subtraction works the same way: find a common denominator and then subtract the numerators, keeping the same denominator.

Here are a few more examples:

Example 1 Compute $\frac{8}{11} + \frac{3}{4}$.

Solution: Multiply each fraction by the other's denominator and then add:

$$\frac{8}{11} \cdot \frac{4}{4} + \frac{3}{4} \cdot \frac{11}{11} = \frac{32}{44} + \frac{33}{44} = \frac{65}{44}.$$

Example 2 Compute $\frac{1}{6} - \frac{5}{8}$.

Solution: Multiply each fraction by the other's denominator and then subtract:

$$\frac{1}{6} \cdot \frac{8}{8} - \frac{5}{8} \cdot \frac{6}{6} = \frac{8}{48} - \frac{30}{48} = -\frac{22}{48}.$$

Exercises

1. Compute the following or explain why it's impossible.
   (a) $\frac{3}{0}$
   (b) $\frac{0}{3}$
2. Compute the following:
   (a) $\frac{2}{3} \cdot \frac{5}{9}$
   (b) $\frac{2}{3} + \frac{5}{9}$
   (c) $\frac{5}{8} - \frac{2}{3}$
   (d) $\frac{\frac{2}{3}}{\frac{7}{10}}$

Answers

1. (a) Impossible. Dividing by 0 is not permitted.

   (b) 0, since 0 divided by anything is 0.

2. (a) $\frac{2}{3} \cdot \frac{5}{9} = \frac{10}{27}$

   (b) $\frac{2}{3} + \frac{5}{9} = \frac{2}{3} \cdot \frac{3}{3} + \frac{5}{9} = \frac{6}{9} + \frac{5}{9} = \frac{11}{9}$

   (c) $\frac{5}{8} - \frac{2}{3} = \frac{5}{8} \cdot \frac{3}{3} - \frac{2}{3} \cdot \frac{8}{8} = \frac{15}{24} - \frac{16}{24} = -\frac{1}{24}$

   (d) $\frac{\frac{2}{3}}{\frac{7}{10}} \cdot \frac{\frac{10}{7}}{\frac{10}{7}} = \frac{\frac{20}{21}}{\frac{70}{70}} = \frac{20}{21}.$