Fractions and algebra

Working with fractions in algebra is quite similar to working with them in ordinary arithmetic. Here are some important techniques.

Reducing fractions (canceling)

In ordinary arithmetic, we often reduce fractions to lowest terms by finding a factor that the top and bottom both have in common and dividing each by it.

For example, to reduce $\frac{12}{16}$ to lowest terms, we notice that both 12 and 16 are divisible by 4. So we divide both 12 and 16 by 4 and the fraction reduces to $\frac{3}{4}$.

We can do the same thing with algebraic expressions. Often we think of this dividing out by a common factor as *canceling* terms. Here are some examples.

Example 1 Simplify $\dfrac{6x^3}{9x^4}$.

Solution: We can divide out or cancel a common factor of 3 and a common factor of x^3 from both the top and bottom to get $\dfrac{2}{3x}$

Example 2 Simplify $\dfrac{x^2(y-1)}{x^3(y-1)^3}$.

Solution: The numerator and denominator both have a common factor of x^2 that we can cancel. That will leave no x terms in the numerator and an x in the denominator. Similarly, the numerator and denominator both have a $y-1$ term. Canceling it leaves nothing in the top and a $(y-1)^2$ term in the bottom. Overall, we have

$$\frac{x^2(y-1)}{x^3(y-1)^3} = \frac{1}{x(y-1)^2}.$$

Example 3 Simplify $\dfrac{x+x^2}{4x}$.

Solution: A useful technique is to first factor and then cancel. That's what we'll do here:

$$\frac{x+x^2}{4x} = \frac{x(1+x)}{4x} = \frac{1+x}{4}.$$

Example 4 Simplify $\dfrac{x^2+2x+1}{x^2+5x+6}$.

Solution: The trick here is we can factor both the numerator and denominator to find a common term to cancel.

$$\frac{x^2+2x+1}{x^2+5x+6} = \frac{(x+1)^2}{(x+1)(x+5)} = \frac{x+1}{x+5}.$$

Example 5 Simplify $\dfrac{2+x^2}{4+x^2}$.

Solution: This is a trick question. The numerator and denominator don't have any factors in common. Trying to cancel the x^2 terms is a *really common mistake*. Don't do it.

The addition is what messes things up. If you have an expression of the form $\frac{AB}{AC}$, where the terms are multiplied together, then the A terms can be canceled. But if it's of the form $\frac{A+B}{A+C}$, the A terms can't be canceled. There's just not an algebra rule that lets you cancel them.

Adding algebraic fractions

When adding fractions, the key is to find a common denominator. The way to do that that always works is to multiply each fraction by the other one's denominator. For instance, to add $\frac{1}{6}+\frac{3}{4}$, we can do the following:

$$\frac{1}{6}\cdot\frac{4}{4}+\frac{3}{4}\cdot\frac{6}{6}=\frac{4}{24}+\frac{18}{24}=\frac{22}{24}.$$

The same exact thing works for algebraic fractions. Here are some examples:

Example 1 Combine $\dfrac{2}{x}+\dfrac{x-1}{3}$ into one fraction.

Solution: Multiply each fraction by the other's denominator to get

$$\frac{2}{x}\cdot\frac{3}{3}+\frac{x-1}{3}\cdot\frac{x}{x}=\frac{6}{3x}+\frac{x(x-1)}{3x}=\frac{6x+x(x-1)}{3x}.$$

Example 2 Combine $\dfrac{x^2}{x-3}+\dfrac{x-1}{x^2+1}$ into one fraction.

Solution: Multiply each fraction by the other's denominator to get

$$\frac{x^2}{x-3}\cdot\frac{x^2+1}{x^2+1}+\frac{x-1}{x^2+1}\cdot\frac{x-3}{x-3}=\frac{x^2(x^2+1)}{(x-3)(x^2+1)}+\frac{(x-1)(x-3)}{(x-3)(x^2+1)}=\frac{x^2(x^2+1)+(x-1)(x-3)}{x^2+1}.$$

Simplifying complex fractions

Recall that division of fractions is done by flipping the denominator and multiplying top and bottom by it, like below:

$$\frac{\frac{2}{5}}{\frac{4}{7}}=\frac{\frac{2}{5}}{\frac{4}{7}}\cdot\frac{\frac{7}{4}}{\frac{7}{4}}=\frac{14}{20}.$$

The same approach works with algebraic expressions. Here are some examples:

Example 1 Simplify $\dfrac{2}{\frac{1}{x}}$.

Solution: Flip the denominator and multiply top and bottom by it to get

$$\frac{\frac{2}{1}}{\frac{1}{x}}\cdot\frac{\frac{x}{1}}{\frac{x}{1}}=\frac{\frac{2x}{1}}{\frac{x}{x}}=2x.$$

The reason for multiplying by the flipped denominator (reciprocal) is that it causes the entire denominator to simplify to 1.

Example 2 Simplify $\dfrac{\frac{x+1}{2x}}{\frac{x^2}{x^2-1}}$

Solution: Multiply top and bottom by the reciprocal of the denominator to get the following:

$$\frac{\frac{x+1}{2x}}{\frac{x^2}{x^2-1}} \cdot \frac{\frac{x^2-1}{x^2}}{\frac{x^2-1}{x^2}} = \frac{(x+1)(x^2-1)}{2x^3}.$$

Remember that when multiplying the denominator by its reciprocal, we just end up with a 1, which we then ignore since anything divided by 1 is itself.

Odds and ends

Breaking up fractions We know that the following is true:

$$\frac{x^2}{5} + \frac{x}{5} = \frac{x^2+x}{5}.$$

It is often handy to use this rule in reverse to break up a fraction into two or more fractions. For example, given $\frac{x+4}{x}$, we can break it up as follows:

$$\frac{x+4}{x} = \frac{x}{x} + \frac{4}{x} = 1 + \frac{4}{x}.$$

This is an occasionally useful algebraic trick. Just be careful though, as you can break up a numerator this way but not a denominator.

Negative exponents Recall that negative exponents correspond to powers in the denominator. For instance, x^{-3} means $\frac{1}{x^3}$.

Having a negative exponent in the denominator means we can turn it to a positive exponent. For instance,

$$\frac{1}{x^{-2}} = \frac{1}{\frac{1}{x^2}} = x^2.$$

We get the x^2 by clearing the complex fraction by multiplying top and bottom by $x^2/1$.

Factoring out numbers It is sometimes handy to move numbers or other things in and out of fractions. Here are a few quick examples:

1. $5\left(\dfrac{x+1}{y}\right) = \dfrac{5(x+1)}{y}$
2. $\dfrac{2x}{x+3} = 2\left(\dfrac{x}{x+3}\right)$
3. $\dfrac{x}{2(x+3)} = \dfrac{1}{2}\left(\dfrac{x}{x+3}\right)$

Exercises

1. Simplify the following.

   (a) $\dfrac{x^3y^3z}{xy^2z}$

   (b) $\dfrac{x^2(x-4)(x+3)}{x^4(x-4)^3}$

   (c) $\dfrac{x^2-1}{x+1}$

   (d) $\dfrac{\frac{x^2+4}{2x}}{\frac{3-x}{3+x}}$

   (e) $\dfrac{2}{x^{-3}}$

2. Add the following fractions.

   (a) $\dfrac{x^2}{2}+\dfrac{1}{x}$

   (b) $\dfrac{x+3}{x+1}+\dfrac{x}{x^2+1}$

3. Is it possible to cancel anything out from the expression $\dfrac{x+1}{x^2+1}$? Explain.

4. Is it true that $3\left(\dfrac{x+1}{x^2}\right)=\dfrac{3x+3}{x^2}$? Explain.

5. Is it true that $\dfrac{x^2+3x+1}{x^2}=1+\dfrac{3}{x}+x^{-2}$? Explain.

Answers

1. (a) Cancel x, y^2 and z from top and bottom to get x^2y

   (b) Cancel x^2 and $(x-4)$ from top and bottom to get $\dfrac{x+3}{x^2(x-4)^2}$

   (c) Factor the numerator into $(x-1)(x+1)$ and cancel $x+1$ from top and bottom. This simplifies the entire expression into $x-1$.

   (d) Simplify the complex fraction as below:

$$\frac{\frac{x^2+4}{2x}}{\frac{3-x}{3+x}}\cdot\frac{\frac{3+x}{3-x}}{\frac{3+x}{3-x}}=\frac{\frac{(x^2+4)(3+x)}{(2x)(3-x)}}{\frac{(3-x)(3+x)}{(3+x)(3-x)}}=\frac{(x^2+4)(3+x)}{(2x)(3-x)}.$$

   (e) Move the x^{-3} from the denominator into the numerator to get $2x^3$.

2. (a) $\dfrac{x^2}{2}+\dfrac{1}{x}=\dfrac{x^2}{2}\cdot\dfrac{x}{x}+\dfrac{1}{x}\cdot\dfrac{2}{2}=\dfrac{x^3+2}{2x}$

   (b) $\dfrac{x+3}{x+1}+\dfrac{x}{x^2+1}=\dfrac{x+3}{x+1}\cdot\dfrac{x^2+1}{x^2+1}+\dfrac{x}{x^2+1}\cdot\dfrac{x+1}{x+1}=\dfrac{(x+3)(x^2+1)+x(x+1)}{(x+1)(x^2+1)}$

3. No. The denominator can't be factored at all, and the individual terms of the numerator and denominator don't have anything in common.

4. Yes. We can think of this as being $\dfrac{3}{1}\cdot\dfrac{x+1}{x^2}$, and multiplying fractions gives $\frac{3x+3}{x^2}$.

5. Yes. We can break it up into three individual fractions as $\frac{x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}$, which simplifies into $1+\frac{3}{x}+x^{-2}$.